Partial Orderings, Lattices, and Boolean Algebra

Partial Orderings

- Definition:

  A binary relation R on a set is called a partial ordering if it is reflexive, antisymmetric, and transitive.

- Example:
  - "refines" is a partial ordering on the set of all the partitions.
- Notation:

  $\leq$ is used as a generic symbol for partial ordering.
  - $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$

    "divides" relation: x divides y if x is a factor of y

    ➡ $2 \leq 6$ and $3 \leq 6$ are true but $5 \leq 6$ is not true.

Partial Orderings

- Definition:

  When R is a partial ordering on a set A, the pair (A, R) is called a partially ordered set or a poset.

- Examples of posets:
  - $(\mathbf{R}, \leq)$
  - (the set of all the partitions, refines)
  - ($\mathbf{Z}^{+}$, divides)
  - $(\wp(A), \subseteq)$

Partial Orderings

- **Theorem**:

  If R is a partial ordering on a set A, then R^c is also a partial ordering on A.

- **Definition**:

  Let R be a partial ordering on a set A and let $X \subseteq A$. The restriction of R on X, denoted R/X, is defined by

  $$R/X = \{ (x, y) \mid x \in X \wedge y \in X \wedge (x, y) \in R\}$$

- Example:

  $A = \mathbf{Z}^+$ $\quad$ $X = \{1, 2, 3, 4, 5, 6, 7\}$

  - divides/$X = \{(1, 1), (1, 2), \ldots, (1, 7), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6), (7,7)\}$

Partial Orderings

- **Theorem**:

  Let R be a partial ordering on a set A and let $X \subseteq A$. Then R/X is a partial ordering on X.

- **Definition**:

  Let R be a partial ordering on a set A. If $a, b \in A$ are such that either $(a, b) \in R$ or $(b, a) \in R$ then a and b are said to be comparable.

- Example:

  $A = \{a, b, c\}$ $\qquad$ $R = \{(a, a), (b, b), (c, c), (a, b)\}$

  - a and b are comparable.
  - a and c are not comparable. b and c are not comparable.

Partial Orderings

- Definition:

  Let R be a partial ordering on a set A such that every pair $a, b \in A$ is comparable. Then R is said to be a linear ordering (total ordering) and (A, R) is said to be a linearly ordered set (totally ordered set) or a chain.

- Definition:

  A relation R on a set A is called a strict partial ordering if it is irreflexive, asymmetric, and transitive.

- Example:

  $A = \{a, b, c\}$ $R = \{(a, a), (b, b), (c, c), (a, b)\}$

  - $R' = \{(a, b)\}$ is a strict partial ordering

Partial Orderings

- Notation:

  $<$ is used as a generic symbol for strict partial ordering.

  $$< \;=\; \{\, (x, y) \mid (x, y) \in \,\leq\, \wedge\, x \neq y\} = \,\leq - E_A$$

- Definition:

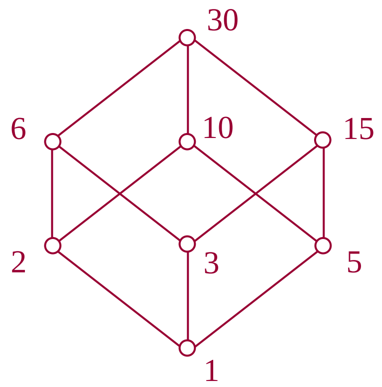
  Let $<$ be a strict partial ordering on a set A. Then the covers relation with respect to $<$ on A, denoted by covers$_<$, is defined as follows:

  covers$_<$ $= \{(x, y) \mid y < x$ and there is no z such that $y < z$ and $z < x\}$

Partial Orderings

- Example:
  - "divides" relation on $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$
  - $< \; = \; \leq - E_A$

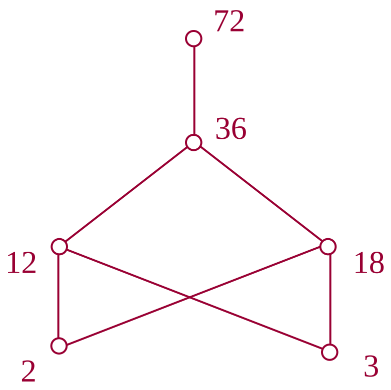
    $= \{(1, 2), \ldots, (1, 30), (2, 6), (2, 10), (2, 30), (3, 6), (3, 15), (3, 30), (5, 10), (5, 15), (5, 30), (6, 30), (15, 30), (10, 30)\}$
  - covers$_<$ $= \{(2, 1), (3, 1), (5, 1), (6, 2), (6, 3), (10, 2), (10, 5), (15, 3), (15, 5), (30, 6), (30, 10), (30, 15)\}$



Hasse Diagram

Partial Orderings

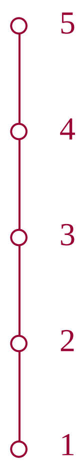
- Example:
  - "divides" relation on $A = \{2, 3, 12, 18, 36, 72\}$
  - $< = \{ (2, 12), (2, 18), (2, 36), (2, 72), (3, 12), (3, 18), (3, 36), (3, 72), (12, 36), (12, 72), (18, 36), (18, 72), (36, 72)\}$
  - covers$_<$ $= \{(12, 2), (12, 3), (18, 2), (18, 3), (36, 12), (36, 18), (72, 36)\}$



We can write ≤ from the Hasse diagram.

Partial Orderings

- Example:
  - “less than or equal to” relation on $A = \{1, 2, 3, 4, 5\}$
  - This relation is a linear ordering.
  - The poset is called a linearly ordered set, totally ordered set, or chain.



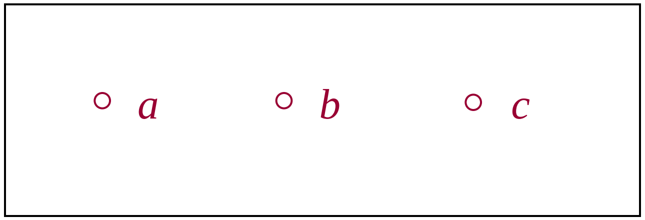
Partial Orderings

- Example:
  - Identity relation E_A on $A = \{a, b, c\}$
  - This relation is
    - reflexive
    - symmetric
    - antisymmetric
    - transitive

equivalence relation

partial ordering

Hasse diagram of E_A:



Bounds

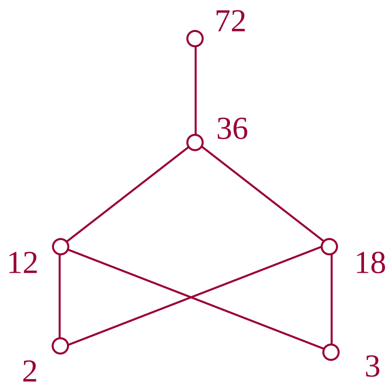
- Definition:

  Let $(A, \leq)$ be a poset and let $X \subseteq A$. Then,

  - $a \in X$ is the greatest element of X if $x \leq a$ for every $x \in X$.
  - $a \in X$ is the least element of X if $a \leq x$ for every $x \in X$.
  - $a \in X$ is the maximal element of X if there is no $x \in X$ such that $a < x$.
  - $a \in X$ is the minimal element of X if there is no $x \in X$ such that $x < a$.

Bounds

- Example:
  - "divides" relation on $A = \{2, 3, 12, 18, 36, 72\}$
  - $X_1 = \{2, 3, 12\}$
    - greatest element of X_1: 12
    - least element of X_1: none
  - $X_2 = \{2, 3, 12, 18\}$
    - greatest element of X_2: none
    - least element of X_2: none



Bounds

- Theorem:

  Let $(A, \leq)$ be a poset and let $X \subseteq A$. Then the greatest (least) element of X if it exists is unique.

- *Proof* :

  Let there be two elements a and b which are the greatest elements of X.

  Then, $a \leq b$ because b is the greatest element of X and $a \in X$.

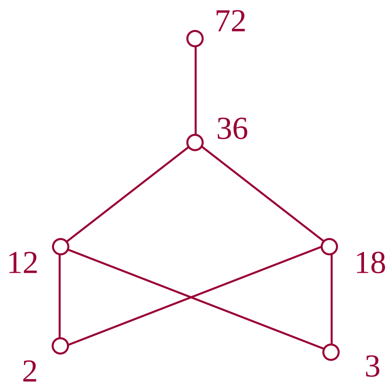
  Similarly, $b \leq a$ because a is the greatest element of X and $b \in X$.

  From $a \leq b$ and $b \leq a$, we conclude $a = b$ because $\leq$ is symmetric.

  □

Bounds

- Example:
  - “divides” relation on $A = \{2, 3, 12, 18, 36, 72\}$
  - $X_1 = \{2, 3, 12\}$
    - maximal element of X_1: 12
    - minimal element of X_1: 2, 3
  - $X_2 = \{2, 3, 12, 18\}$
    - maximal element of X_2: 12, 18
    - minimal element of X_2: 2, 3



Bounds

- **Theorem**:

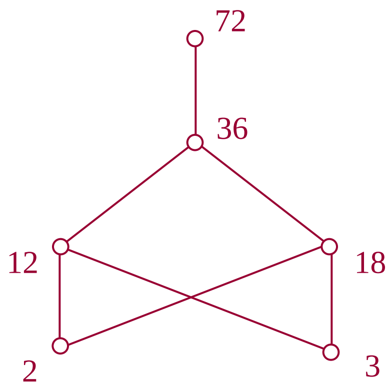
  Let $(A, \leq)$ be a poset and let $X \subseteq A$. If $a \in X$ is the unique maximal (minimal) element of X then a is the greatest (least) element of X.

- **Definition**:

  Let $(A, \leq)$ be a poset and let $X \subseteq A$. Then,
  - $a \in A$ is the upper bound of X if $x \leq a$ for every $x \in X$.
  - $a \in A$ is the lower bound of X if $a \leq x$ for every $x \in X$.

Bounds

- Example:
  - "divides" relation on $A = \{2, 3, 12, 18, 36, 72\}$
  - $X = \{12, 18, 36\}$
    - greatest element: 36
    - least element: none
    - maximal element: 36
    - minimal element: $12, 18$
    - upper bound: $36, 72$
    - lower bound: $2, 3$



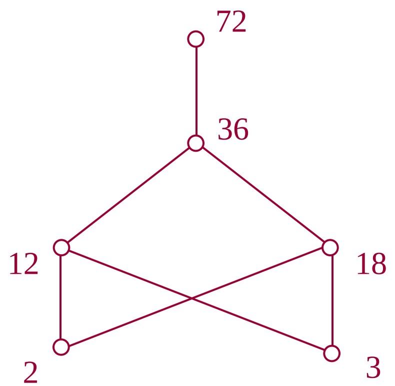
Bounds

- Definition:

  Let $(A, \leq)$ be a poset and let $X \subseteq A$. Then,

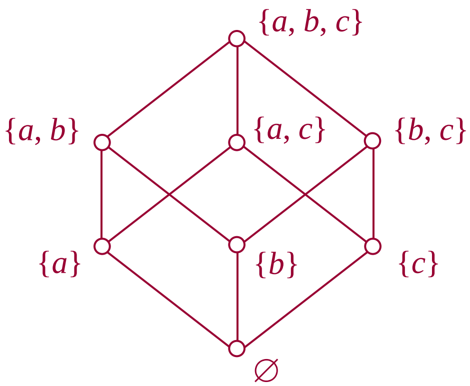
  - The least element of the set of upper bounds of X is called the least upper bound (LUB, supremum) of X.
  - The greatest element of the set of lower bounds of X is called the greatest lower bound (GLB, infimum) of X.

- Example:
  - "divides" relation on $A = \{2, 3, 12, 18, 36, 72\}$
  - $X = \{12, 18, 36\}$
    - LUB of X: 36
    - GLB of X: none



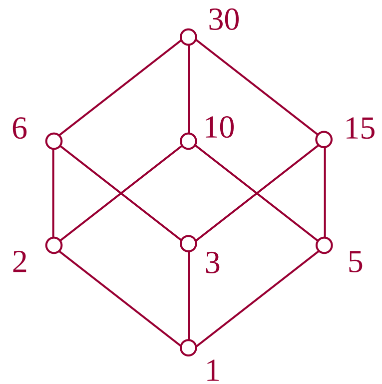
Bounds

- Example:
  - Consider the poset $(\wp(A), \subseteq)$ where $A = \{a, b, c\}$.
  - Let $X_i, X_j \in \wp(A)$. Then,
    - LUB of $\{X_i, X_j\} = X_i \cup X_j$
    - GLB of $\{X_i, X_j\} = X_i \cap X_j$



Bounds

- Example:
  - Consider the poset $(A, \text{divides})$ where $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$.
  - Let $a, b \in A$. Then,
    - LUB of $\{a, b\}$ = LCM (Least Common Multiple) of a and b
    - GLB of $\{a, b\}$ = GCD (Greatest Common Divisor) of a and b



Isomorphism

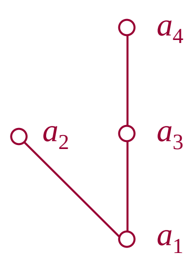
- Definition:

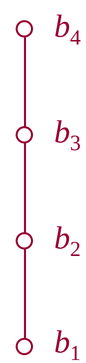
  Let $(A, \leq)$ and $(B, \leq')$ be two posets and let $f : A \rightarrow B$. The function f is said to be order preserving relative to $\leq$ and $\leq'$ if and only if for every $a, b \in A$ if $a \leq b$ then $f(a) \leq' f(b)$.

Isomorphism

- Example:
  - $f : A \to B$

    $A = \{a_1, a_2, a_3, a_4\}$ $B = \{b_1, b_2, b_3, b_4\}$





  - $f(a_i) = b_i \ (1 \leq i \leq 4)$ is order preserving.

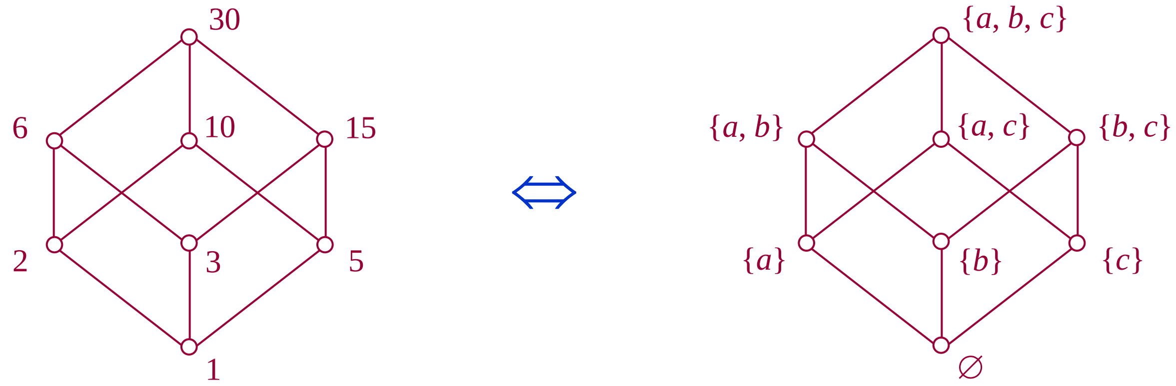
    $a_i \leq a_j \ \to \ f(a_i) = b_i \leq' b_j = f(a_j)$ for all i, j.
  - $f^{-1}: B \to A$ is not order preserving.

Isomorphism

- **Definition**:

  Let $(A, \leq)$ and $(B, \leq')$ be two posets and let $f : A \to B$. If both f and f^{-1} is order preserving, then f is said to be an order isomorphism (or just isomorphism) between $(A, \leq)$ and $(B, \leq')$ and the posets are said to be order-isomorphic (or just isomorphic).

- Example:

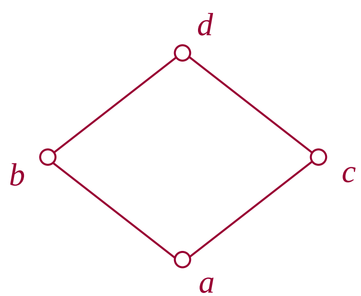


Lattices

- Definition:

  A poset $(A, \leq)$ is said to be a lattice if for every $a, b \in A$ there is a LUB and a GLB.

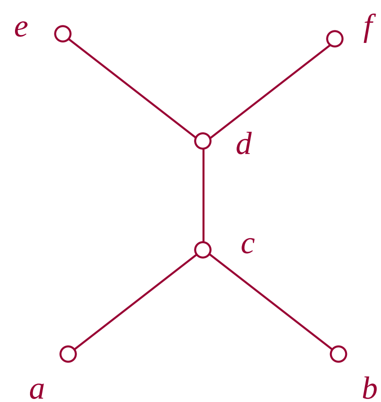
- Examples:



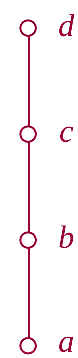
Lattice

GLB$(b, c) = a$ LUB$(b, c) = d$

GLB$(a, b) = a$ LUB$(a, b) = b$



Not a lattice



Lattice

Not a lattice

identity relation

Lattices

- Operation:
  - An n-ary operation on a set A is a function.

$$f : \underbrace{A \times A \times \cdots \times A}_{n} \to A$$

  - Binary operation

$$f : A \times A \to A$$

  - On a lattice, GLB and LUB are binary operations.
    - GLB$(a, b) = a * b$
    - LUB$(a, b) = a + b$

Lattices

- Theorem:

  If $(A, \leq)$ is a lattice, then for any $x, y \in A$

  1. $x * y = x$ iff $x \leq y$
  2. $x + y = y$ iff $x \leq y$

- *Proof* of 1

  (if part)

  Assume $x \leq y$.

  Since $x \leq x$ and $x \leq y$, x is a lower bound of x and y.

  We know $x * y$ is also a lower bound and it is the greatest lower bound.

  Thus $x \leq x * y$.

Lattices

- *Proof of 1*:

    But $x * y$ is a lower bound of x and y. Thus $x * y \leq x$.

    From $x \leq x * y$ and $x * y \leq x$, we conclude that $x * y = x$.

(only if part)

Assume $x * y = x$.

We know $x * y$ is the greatest lower bound of x and y.

Thus $x * y \leq y$.

Since $x * y = x$, we conclude that $x \leq y$.

□

Lattices

- **Theorem**:

  Let $(A, \leq)$ is a lattice. Then for every x, y , $z \in A$ the following are true:

  1. $x * x = x$ $\quad$ $x + x = x$ $\quad$ idempotent laws
  2. $x * y = y * x$ $\quad$ $x + y = y + x$ $\quad$ commutative laws
  3. $x * (y * z) = (x * y) * z$ $\quad$ $x + (y + z) = (x + y) + z$ $\quad$ associative laws
  4. $x * (x + y) = x$ $\quad$ $x + (x * y) = x$ $\quad$ absorption laws

- *Proof* of 1

  Using the previous theorem and the fact that $x \leq x$, we can easily show that $x * x = x$ and $x + x = x$. $\square$

Lattices

- *Proof of 4*:

  $x \leq x + y$ because $x + y$ is the least upper bound of x and y.

  Again, using the previous theorem, $x * (x + y) = x$.

  □

Algebra

$$(\overset{\text{set}}{\downarrow}A, \overbrace{\underbrace{O_{11}, O_{12}, \ldots, O_{1n_1}}_{\text{unary operations}}, \underbrace{O_{21}, O_{22}, \ldots, O_{2n_2}}_{\text{binary operations}}, \ldots, \underbrace{O_{k1}, O_{k2}, \ldots, O_{kn_k}}_{k\text{-ary operations}}}^{\text{operations on } A})$$

Algebra

- **Theorem**: Let $(A, *, +)$ be an algebra such that the following four pairs of laws are satisfied:

  1. $x * x = x$ $\quad$ $x + x = x$
  2. $x * y = y * x$ $\quad$ $x + y = y + x$
  3. $x * (y * z) = (x * y) * z$ $\quad$ $x + (y + z) = (x + y) + z$
  4. $x * (x + y) = x$ $\quad$ $x + (x * y) = x$

  Then $(A, \leq)$ is a lattice if, for every $x, y \in A$, $x \leq y$ when $x * y = x$ and/or $x + y = y$.

- *Proof* :

  First, we want to show that $\leq$ is a partial ordering.

Algebra

- *Proof* :

Since $x * x = x$ for every $x \in A$, we have $x \leq x$ and so $\leq$ is reflexive.

If $x \leq y$ and $y \leq x$, then $x * y = x$ and $y * x = y$.

But $x * y = y * x$ is given, and so $x = y$. Thus $\leq$ is antisymmetric.

If $x \leq y$ and $y \leq z$, then $x * y = x$ and $y * z = y$.

Substituting $y * z$ for y in $x * y = x$, we get $x * (y * z) = x$.

By applying 3, we get $(x * y) * z = x$.

Substituting x for $x * y$, we get $x * z = x$.

Thus $x \leq z$ and so $\leq$ is transitive.

Since $\leq$ is reflexive, antisymmetric, and transitive, $\leq$ is a partial ordering.

Algebra

- *Proof* :

  Now we have to show that there exists a g.l.b. and a l.u.b. of x and y for every $x, y \in A$.

  Since $x * (x + y) = x$ for every $x, y \in A$, we have $x \leq x + y$.

  Similarly since $y * (y + x) = y$, we have $y \leq y + x = x + y$.

  From $x \leq x + y$ and $y \leq x + y$, we conclude that $x + y$ is an upper bound of x and y.

  If $x + y$ is the only upper bound, then it is the l.u.b. of x and y.

  If not, suppose there is another upper bound, say z, of x and y.

  In this case, $x \leq z$ and $y \leq z$, and thus $x + z = z$ and $y + z = z$.

Algebra

- *Proof* :

  Substituting $y + z$ for z in the left hand side of $x + z = z$, we get $x + (y + z) = z$.

  Using 3, we get $(x + y) + z = z$.

  Hence, $x + y \leq z$ and thus $x + y$ is the l.u.b. of x and y.

  We can similarly show that $x * y$ is the g.l.b. of x and y.

  Therefore, $(A, \leq)$ is a lattice.

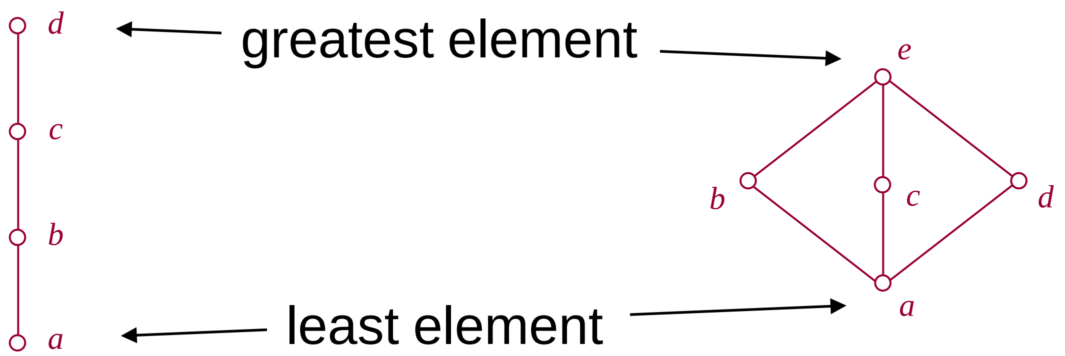
  □

Boolean Lattice and Boolean Algebra

- Definition:

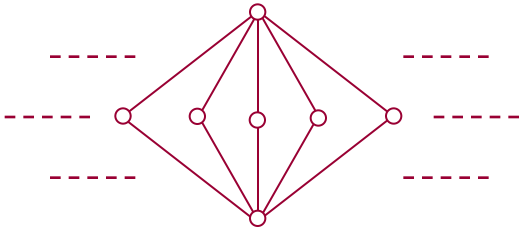
  A lattice $(A, \leq)$ is said to be a bounded lattice if the set A has a greatest element and a least element.

- Example:



Boolean Lattice and Boolean Algebra

- Note:
  - In a lattice,
    - the greatest element is usually denoted by '1', and
    - the least element is usually denoted by '0'.
  - For all x,
    - $0 \le x \rightarrow 0 * x = 0$
    - $x \le 1 \rightarrow x + 1 = 1$



- Theorem: If $(A, \le)$ is a finite lattice then it is a bounded lattice.

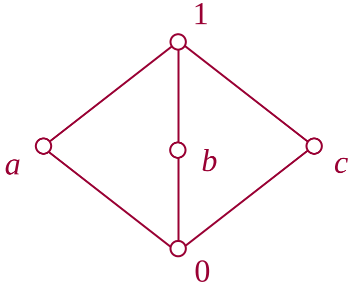
  (The converse is not necessarily true.)

Boolean Lattice and Boolean Algebra

- Definition:

  A bounded lattice $(A, \leq)$ is said to be a complemented lattice if for every $x \in A$ there is an $\overline{x} \in A$ such that $x * \overline{x} = 0$ and $x + \overline{x} = 1$.

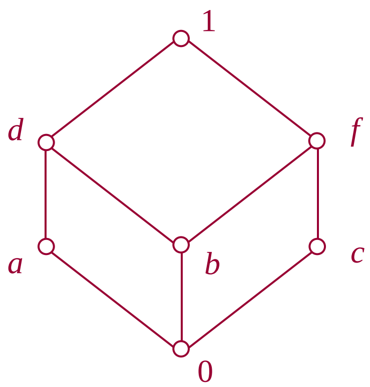
- Example:
  - Let x be a complement of a. Then,
    - $a * x = 0 \to x = b, c, 0$
    - $a + x = 1 \to x = b, c, 1$
    - ➡ $\overline{a} = b$ or $\overline{a} = c$



A complemented lattice

Boolean Lattice and Boolean Algebra

- Example:
  - $b * x = 0 \rightarrow x = a, c, 0$
  - $b + x = 1 \rightarrow x = 1$
  - ➡ There is no $\bar{b}$.



Not a complemented lattice

Boolean Lattice and Boolean Algebra

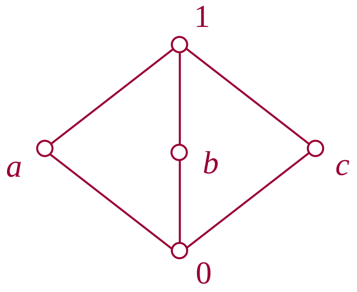
- **Definition**:

  A bounded lattice $(A, \leq)$ is said to be a distributive lattice if for every $x, y, z \in A$ the following are satisfied:

  1. $x * (y + z) = (x * y) + (x * z)$, and
  2. $x + (y * z) = (x + y) * (x + z)$

- Example:
  - $a * (b + c) = a * 1 = a$
  - $(a * b) + (a * c) = 0 + 0 = 0$

  ➡ Not a distributive lattice



Boolean Lattice and Boolean Algebra

poset

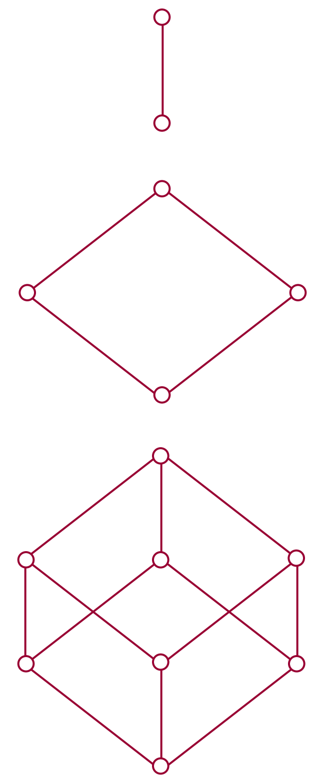
lattice

bounded lattice

complemented lattice

distributive lattice

Boolean lattice



Boolean lattices

Boolean Lattice and Boolean Algebra

- Lattice and algebra:
  - From a lattice $(A, \leq)$ we can define an algebra $(A, *, +)$, and vice versa.

lattice $(A, \leq)$ $\cdots$ $(A, *, +)$ algebra

Boolean lattice $(A, \leq)$ $\cdots$ $(A, *, +, \bar{\ }, 0, 1)$ Boolean algebra

binary operations unary operation constants

Boolean Lattice and Boolean Algebra

- Boolean algebra:
  - The following four laws are satisfied: (lattice)
    - idempotent law
    - commutative law
    - associative law
    - absorption law
  - $x + 0 = x$ and $x * 1 = x$ for every $x \in A$. (bounded lattice)
  - For every $x \in A$ there is $\overline{x} \in A$ such that $x * \overline{x} = 0$ and $x + \overline{x} = 1$. (complemented lattice)
  - For every $x, y, z \in A$ we have $x * (y + z) = (x * y) + (x * z)$ and $x + (y * z) = (x + y) * (x + z)$ (distributive lattice)