Equivalence Relations & Partitions

Equivalence Relations

- Definition:

  A binary relation R on a set is called an equivalence relation if it is reflexive, symmetric, and transitive.

- Example:

  W: set of all words in English dictionary

  R: “has the same first letter as” relation

  Claim: R is an equivalence relation on the set A.

  $$(\forall w)\ \ w \in W \rightarrow (w, w) \in R$$

  $$(\forall w_1)(\forall w_2)\ \ (w_1, w_2) \in R \rightarrow (w_2, w_1) \in R$$

  $$(\forall w_1)(\forall w_2)(\forall w_3)\ \ (w_1, w_2) \in R \wedge (w_2, w_3) \in R \rightarrow (w_1, w_3) \in R$$

Equivalence Relations

- More examples:
  - E_A for a set A is an equivalence relation.
  - $A \times A$ is an equivalence relation.
  - How many relations on A are equivalence relations?
  - $R_k = \{(x, y) \mid x, y \in \mathbf{Z},\ x - y = n \cdot k,\ k \in \mathbf{Z}^+,\ n \in \mathbf{Z}\}$

    We say "x and y are equivalent modulo k."

    When $k = 3$,

    $(7, 7) \in R_3$

    $(7, 4) \in R_3\ (7 - 4 = 1 \cdot 3) \rightarrow (4, 7) \in R_3\ (4 - 7 = (-1) \cdot 3)$

    $(4, 10) \in R_3 \wedge (10, 19) \in R_3 \rightarrow (4, 19) \in R_3$

Equivalence Relations

- Theorem:

  If R_1 and R_2 are two equivalence relations on a set A then $R_1 \cap R_2$ is an equivalence relation.

- *Proof* :

  We need to show that $R_1 \cap R_2$ is reflexive, symmetric, and transitive.

(Reflexive)

Let $a \in A$. We must show that $(a, a) \in R_1 \cap R_2$.

Since R_1 and R_2 are equivalence relations, they must be reflexive and so $(a, a) \in R_1$ and $(a, a) \in R_2$.

Equivalence Relations

- *Proof* :

Therefore, $(a, a) \in R_1 \cap R_2$ and $R_1 \cap R_2$ is reflexive.

(Symmetric)

Let $(a, b) \in R_1 \cap R_2$. Then, $(a, b) \in R_1$ and $(a, b) \in R_2$.

Since R_1 and R_2 are symmetric $(b, a) \in R_1$ and $(b, a) \in R_2$.

Therefore, $(b, a) \in R_1 \cap R_2$ and $R_1 \cap R_2$ is symmetric.

(Transitive)

Left as an exercise.

□

Equivalence Relations

- A counter example for $R_1 \cup R_2$:
  - $A = \{a, b, c\}$
  - $R_1 = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$ is an equivalence relation
  - $R_2 = \{(a, a), (b, b), (c, c), (a, c), (c, a)\}$ is an equivalence relation
  - ➡ $R_1 \cup R_2 = \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a)\}$

    is not transitive, and so is not an equivalence relation.
- Theorem:

  Let R be a non-empty relation on a set A. Then,
  - $tsr(R)$ is an equivalence relation.
  - If R' is any equivalence relation such that $R \subseteq R'$, then $tsr(R) \subseteq R'$.

Equivalence Classes

- Definition:

  Let R be an equivalence relation on a set A. For each $x \in A$, the equivalence class of x with respect to R, denoted by $[x]_R$ is defined by

  $$[x]_R = \{ y \in A \mid (x, y) \in R \}.$$

- Example:

  W: set of all words in English dictionary

  R: “has the same first letter as” relation

  $[\mathrm{dog}]_R$: set of all the words that start with the letter ‘d’

Equivalence Classes

- Theorem:

  Let R be an equivalence relation on a set A. Then,

  $$[a]_R = [b]_R \text{ iff } (a, b) \in R$$

  (Ex: $[\mathrm{dog}]_R = [\mathrm{dummy}]_R$)

- *Proof* :

  (if part)

  Assume $(a, b) \in R$.

  Let $x \in [a]_R$.

  Then, $(a, x) \in R$ and $(x, a) \in R$ because R is an equivalence relation and thus symmetric.

Equivalence Classes

- *Proof* :

  From $(x, a) \in R$ and $(a, b) \in R$, we have $(x, b) \in R$ because R is an equivalence relation and thus transitive.

  Then, $(b, x) \in R$ because R is an equivalence relation and thus symmetric. So, we have $x \in [b]_R$.

  Therefore, $[a]_R \subseteq [b]_R$.

  We can similarly show that $[b]_R \subseteq [a]_R$.

  Therefore, $[a]_R = [b]_R$.

Equivalence Classes

- *Proof :*

  (only if part)

  Assume $[a]_R = [b]_R$.

  Let $x \in [a]_R$. Then, $x \in [b]_R$.

  Then, $(a, x) \in R$ and $(b, x) \in R$.

  Since $(b, x) \in R$ and R is symmetric, we have $(x, b) \in R$.

  From $(a, x) \in R$ and $(x, b) \in R$, $(a, b) \in R$ because R is transitive.

  □

Equivalence Classes

- **Theorem**:

  Let R be an equivalence relation on a set A. Then,

  1. $a \in [a]_R$
  2. Either $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \varnothing$ but not both
  3. $\bigcup_{a \in A} [a]_R = A$

- Example: English dictionary
  - $\text{dog} \in [\text{dog}]_R$
  - $[\text{dog}]_R = [\text{dummy}]_R$ $\qquad$ $[\text{dog}]_R \cap [\text{cat}]_R = \varnothing$
  - $\bigcup_{a \in A} [a]_R = W$ (set of all words)

Equivalence Classes

- *Proof of* 2 :

  Suppose $[a]_R \neq [b]_R$ and $[a]_R \cap [b]_R \neq \varnothing$.

  Let x be an element of the nonempty set $[a]_R \cap [b]_R$.

  Then, $x \in [a]_R$ and $x \in [b]_R$ and so $(a, x) \in R$ and $(b, x) \in R$.

  But, $(x, b) \in R$ because R is symmetric.

  From $(a, x) \in R$ and $(x, b) \in R$ we get $(a, b) \in R$ because R is transitive.

  Then, $[a]_R = [b]_R$, which is a contradiction. □

Equivalence Classes

- Definition:

  Let R be an equivalence relation on a set A. The quotient set of A modulo R, denoted by A/R, is defined by

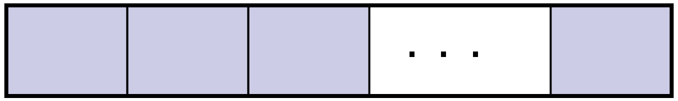
  $$A/R = \{[x]_R \mid x \in A\}$$

- Example:
  - W: English dictionary R: has the same first letter as

  $W/R = \{\{$all words starting with $a\}$, $\{$all words starting with $b\}, \ldots$

  $\ldots, \{$all words starting with $z\}\}$

W/R:



Equivalence Classes

- Example:
  - $A = \{a, b, c\}$
  - $R = \{(a, a), (b, b), (c, c), (a, c), (c, a)\}$
  - $A/R = \{\{a, c\}, \{b\}\}$

Partitions

- Definition:

  Let A be a nonempty set and let π be a collection of nonempty subsets of A such that

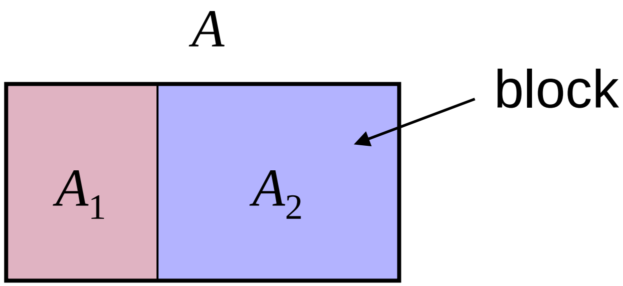
  1. If $X, Y \in \pi$ and $X \neq Y$ then $X \cap Y = \varnothing$
  2. $\bigcup\limits_{X \in \pi} X = A$

  then π is called a partition of A.

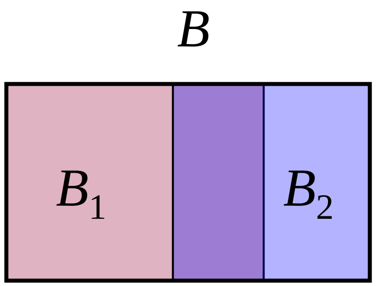
- Note:
  - $\varnothing \notin \pi$ and $\pi \subset \wp(A)$
  - If only 2 holds, then it is called a cover.

Partitions

- Example:



$\{A_1, A_2\}$: partition



$\{B_1, B_2\}$: cover, not a partition

- Definition:
  - ☐ Let π be a partition on a set A. If π is finite then $|\pi|$ is called the rank of the partition A.

Partitions

- Theorem:

  If R is an equivalence relation on a set A, then A/R is a partition of A.

- *Proof* :

  Let $X, Y \in A/R$.

  Then, X and Y are equivalence classes.

  We know that (1) either $X = Y$ or $X \cap Y = \varnothing$ and (2) $\bigcup_{X \in A/R} X = A$.

  This implies that A/R is a partition of A.

  □

Partitions

- Definition:

  Let π be a partition on a set A. The relation induced by the partition π, denoted by R_π, is defined as

  $$R_\pi = \{(x, y) \mid \exists S\,(S \in \pi \land x \in S \land y \in S)\}$$

- Example:
  - $A = \{a, b, c, d, e, f, g\}$
  - $\pi = \{\{a, b, c\}, \{d, e\}, \{f, g\}\}$
  - $R_\pi = \{(a, a), (b, b), \ldots, (g, g), (a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (d, e), (e, d), (f, g), (g, f)\}$

Partitions

- **Theorem**:

  Let R_π be the relation induced by a partition π on a nonempty set A. Then,

  1. R_π is an equivalence relation.
  2. $A/R_\pi = \pi$.

Partitions

$$\begin{tikzcd}
 & R \arrow[r] & A/R \\
A & & \\
 & \pi \arrow[r] & R_\pi
\end{tikzcd}$$

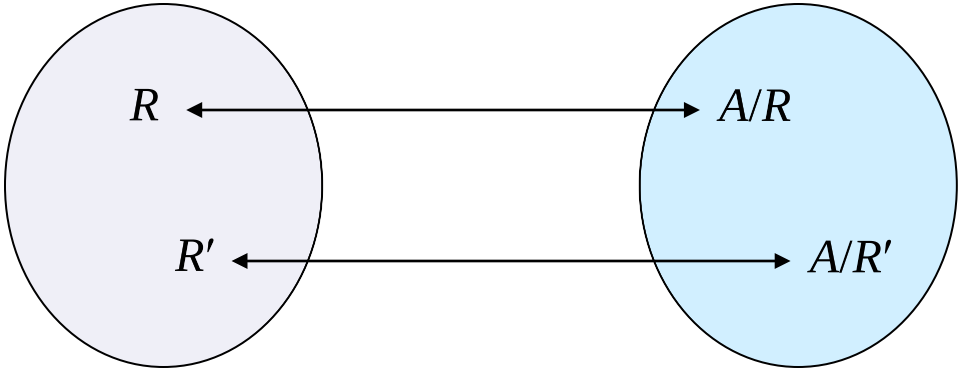
$R = R_\pi$ iff $\pi = A/R$

- Theorem:

  Let R be an equivalence relation on a nonempty set A and let π be a partition on the set. Then,

$$R = R_\pi \text{ iff } \pi = A/R$$

Partitions



set of all equivalence relations on A

set of all partitions on A

- **Theorem**:

  There exists a one-to-one correspondence between the set of all equivalence relations on a nonempty set A and the set of all partitions on A.

Partitions

- Definition:

  Let π and π' be two partitions on a nonempty set A. Then π' is said to refine π (π' is a refinement of π) if every block of π' is a subset of some block of π.

- Example:
  - ☐ $A = \{a, b, c, d, e, f, g\}$
  - ☐ $\pi = \{\{a, b, c\}, \{d, e\}, \{f, g\}\}$
  - ☐ $\pi' = \{\{a, b\}, \{c\}, \{d, e\}, \{f, g\}\}$
  - ➡ π' is a refinement of π

Partitions

- Note:
  - $\pi_0 = \{\{a, b, c, d, e, f, g\}\}$

    Every partition is a refinement of π_0.
  - $\pi_\infty = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}\}$

    π_∞ is a refinement of every partition.
  - A partition refines itself.

- Definition:

  If a partition π' refines a partition π and if $\pi' \neq \pi$, then π' is called a proper refinement of π.

Partitions

- Definitions:
  - ☐ Let π_1 and π_2 be two partitions on a nonempty set A. $\pi_1 \cdot \pi_2$ is a partition on A that refines both π_1 and π_2 and if π' is another partition that refines π_1 and π_2 then π' refines $\pi_1 \cdot \pi_2$.
  - ☐ Let π_1 and π_2 be two partitions on a nonempty set A. $\pi_1 + \pi_2$ is a partition on A that is refined by both π_1 and π_2 and if π' is another partition that is refined by π_1 and π_2 then π' is refined by $\pi_1 + \pi_2$.

Partitions

- Example:
  - $\pi_1 = \{\{a, b, c\}, \{d, e\}, \{f, g\}\}$
  - $\pi_2 = \{\{a, b, c, d\}, \{e\}, \{f, g\}\}$
  - $\pi_1 \cdot \pi_2 = \{\{a, b, c\}, \{d\}, \{e\}, \{f, g\}\}$
  - $\pi_3 = \{\{a, b\}, \{c\}, \{d\}, \{e\}, \{f, g\}\}$

    $\pi_3 \neq \pi_1 \cdot \pi_2$

    π_3 is a proper refinement of $\pi_1 \cdot \pi_2$.
  - $\pi_1 + \pi_2 = \{\{a, b, c, d, e\}, \{f, g\}\}$

Partitions

- Theorem:
  - ☐ The relation “refines” on the set of all the partitions on a nonempty set is reflexive, antisymmetric, and transitive.
- Theorem:
  - ☐ Let π_1 and π_2 be two partitions on a nonempty set A. Then
    - $\pi_1 \cdot \pi_2 = A/(R_{\pi_1} \cap R_{\pi_2})$
    - $\pi_1 + \pi_2 = A/t(R_{\pi_1} \cup R_{\pi_2})$.
- Corollary:

  Given two partitions π_1 and π_2 on a nonempty set A, there is a unique $\pi_1 \cdot \pi_2$ and a unique $\pi_1 + \pi_2$.