Relations and Functions

– Part B –

Functions

- Definition:

  A function f from a set A to a set B, denoted by $f : A \to B$, is a relation between A and B in which for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$.

- Example:

  $A = \{a, b, c\}, \; B = \{\alpha, \beta, \gamma, \delta\}$

  $R_1 = \{(a, \alpha), (a, \delta), (b, \gamma)\}$ is not a function.

  $R_2 = \{(a, \alpha), (b, \alpha), (c, \delta)\}$ is a function.

Functions

- Note:
  - For a function $f: A \to B$, $(c, \delta) \in f$ is also written as $f(c) = \delta$ where c is called the argument and δ is called the value.
  - As a relation, $(c, \delta) \in f$ is sometimes written as $c\, f\, \delta$.
  - Domain $\mathcal{D}(f) = A$ and range $\mathcal{R}(f) \subseteq B$ (B : codomain of f).
  - The order of function composition is the reverse of relations.

    Let $A = \{a, b, c\}$, $B = \{\alpha, \beta\}$, and $C = \{x, y, z\}$.

    Let $f = \{(a, \beta), (b, \beta), (c, \alpha)\}$ and $g = \{(\alpha, x), (\beta, z)\}$.

    Then, $g \circ f = \{(a, z), (b, z), (c, x)\}$. (not written as $f \circ g$)

    $g \circ f(a) = g(f(a)) = z$.

Functions

- Commutative Diagram

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C \end{tikzcd}$$

Functions

- Theorem:

  If $f : A \to B$ and $g : B \to C$ then $g \circ f : A \to C$.

- *Proof* :

  $g \circ f$ is obviously a relation between A and C.

  To prove that for every $a \in A$ there is a unique $c \in C$ such that $(a, c) \in g \circ f$.

  For every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$ because f is a function.

  But for $b \in B$ there is a unique $c \in C$ such that $(b, c) \in g$ because g is a function.

  Therefore, $g \circ f : A \to C$. □

Functions

- Theorem:

  Let $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then $(h \circ g) \circ f = h \circ (g \circ f)$.

  (i.e., function composition is associative.)

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[dr, "h \circ g"] & C \arrow[d, "h"] \\
A \arrow[u, "f"] \arrow[ur, "g \circ f"] \arrow[r, "(h \circ g) \circ f = h \circ (g \circ f)"'] & D
\end{tikzcd}
$$

Functions

- Notation:

  B^A denote the set of all functions from A to B.

  Suppose $|A| = n$ and $|B| = m$. Then, $|B^A| = m^n$.

- Definitions: Let $f : A \to B$.
  - f is said to be surjective if $\mathcal{R}(f) = B$.
  - f is said to be injective if for every $(a, b) \in f$ and $(a', b) \in f$, $a = a'$.
  - If f is both surjective and injective then it is called bijective.

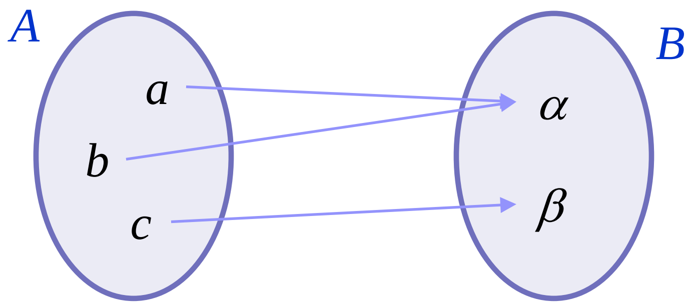
Functions

- Terminologies:

is a surjection ⋯ is surjective ⋯ is onto

is an injection ⋯ is injective ⋯ is one-to-one

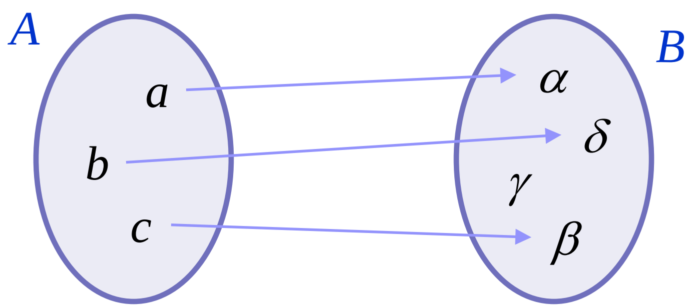
is a bijection ⋯ is bijective ⋯ is one-to-one and onto



surjective
not injective

Functions

- Terminologies:



injective
not surjective

  - Note that for a function $f : A \to B$ to be bijective, A and B should have the same cardinality.

Functions

- Theorem: Let $f : A \to B$ and $g : B \to C$.

  (1) If f and g are surjective then $g \circ f$ is surjective.

  (2) If f and g are injective then $g \circ f$ is injective.

  (3) If f and g are bijective then $g \circ f$ is bijective.

- *Proof* of (2)

  Note that $g \circ f : A \to C$.

  Let (a, c) and (a', c) be elements of $g \circ f$.

  Since $(a, c) \in g \circ f$, there must be a $b \in B$ such that $(a, b) \in f$ and $(b, c) \in g$.

Functions

- *Proof* of (2)

  Similarly for $(a', c) \in g \circ f$, there must be a $b' \in B$ such that $(a', b') \in f$ and $(b', c) \in g$.

  But $b = b'$ because $(b, c) \in g$, $(b', c) \in g$, and g is injective.

  Since $(a, b) \in f$, $(a', b') \in f$, $b = b'$, and f is injective, we get $a = a'$.

  □

- **Theorem:** Let $f : A \to B$ and $g : B \to C$.
  - If $g \circ f$ is surjective then g is surjective.
  - If $g \circ f$ is injective then f is injective.
  - If $g \circ f$ is bijective then g is surjective and f is injective.

Functions

- Definition:

  The function $1_A : A \to A$, denoted by $1_A(a) = a$ for all $a \in A$, is called the identity function for A.

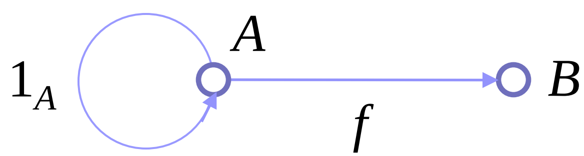
- Example:

  If $A = \{a, b, c\}$ then the identity function 1_A for A is

  $$\{(a, a), (b, b), (c, c)\}.$$

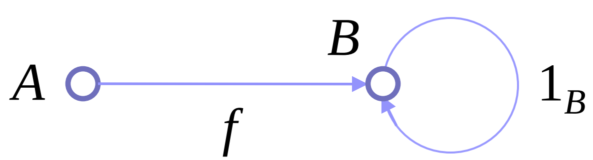
Functions

- Right and left identities: Let $f : A \to B$.
  - $f \circ 1_A = f$, where 1_A is called the right identity function of f.



($1_A \circ f = f$? $\cdots$ meaningless if $A \neq B$)

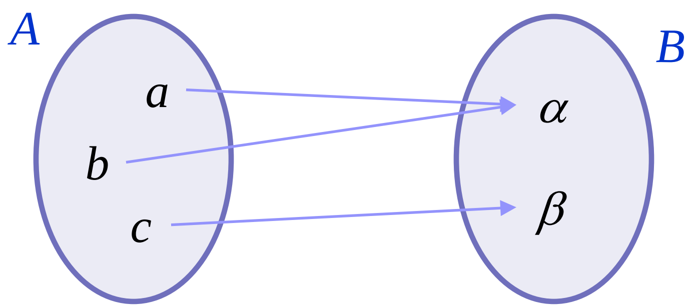
  - $1_B \circ f = f$, where 1_B is called the left identity function of f.



Functions

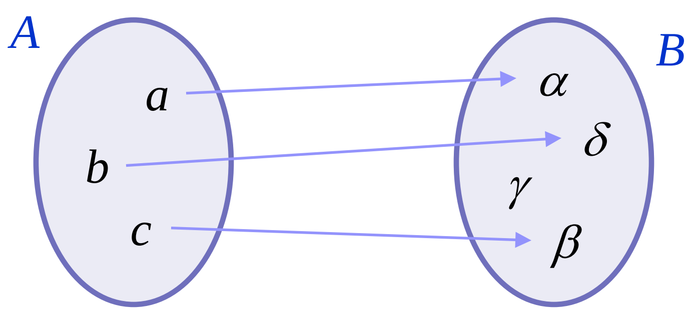
- Inverse function:

Let $f : A \to B$. Then the converse of f is $f^c = \{(y, x) \mid (x, y) \in f\}$.



$f_1 : A \to B$

f_1^c is not a function

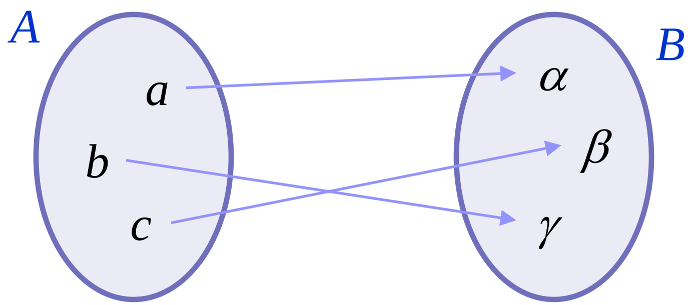


$f_2 : A \to B$

f_2^c is not a function

Functions

- Inverse function:



$f_3 : A \rightarrow B$

$f_3{}^c$ is a function

The converse of a function is not necessarily a function.

If there exists a converse which is a function, it is called the inverse of f, and is denoted by f^{-1}.

Functions

- Right and left inverses: Let $f : A \to B$ and $g : B \to A$.
  - If $f \circ g = 1_B$, then g is called the right inverse of f.

$$\begin{tikzcd} B \arrow[r, bend left, "g"] & A \arrow[l, bend left, "f"] \end{tikzcd}$$

  - If $g \circ f = 1_A$, then g is called the left inverse of f.

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$$

Functions

- **Theorem:** Let $f: A \to B$.

  (1) f has a left inverse if and only if f is injective.

  (2) f has a right inverse if and only if f is surjective.

  (3) f has a left and right inverse if and only if f is bijective.

  (4) If f is bijective then the left inverse of f is equal to the right inverse of f.

- *Proof* of (1)

  (if part): f has a left inverse if f is injective.

  Assume that f is injective.

  Let $g: B \to A$ be defined as follows.

Functions

- *Proof* of (1)

For $b \in B$,

$$g(b) = \begin{cases} a & \text{if } b \in \mathcal{R}(f) \text{ and } f(a) = b \\ c & \text{otherwise, where } c \text{ is a unique element of } A \end{cases}$$

Obviously g is a relation and $\mathcal{D}(g) = B$.

Let $(x, y) \in g$ and $(x, z) \in g$.

If $x \notin \mathcal{R}(f)$, then $y = z$ by the lower part of g's definition.

If $x \in \mathcal{R}(f)$, then $f(y) = x$ and $f(z) = x$.

But since f is injective, $y = z$.

Hence, when $(x, y) \in g$ and $(x, z) \in g$, $y = z$.

Thus, g is a function.

Functions

- *Proof* of (1)

  Note that $g \circ f : A \to A$.

  Let $a \in A$.

  Then $g \circ f(a) = g(f(a)) = a$ by the top part of g's definition.

  Hence, $g \circ f = 1_A$.

  Therefore, g is the left inverse of f.

  (only if part): f has a left inverse only if f is injective.

  Assume that f has a left inverse.

  Let $g : B \to A$ be a left inverse of f, i.e., $g \circ f = 1_A$.

  We want to prove that f is injective.

Functions

- *Proof* of (1)

  Assuming $(x, y) \in f$ and $(z, y) \in f$, we have to show $x = z$.

  $$x = 1_A(x) = g \circ f(x) = g(f(x)) = g(y) = g(f(z)) = g \circ f(z) = 1_A(z) = z$$

  Hence, f is injective.

  □

- *Proof* of (4)

  Since f is surjective, it has a right inverse.

  Let that inverse be $f_R : B \to A$.

  Since f is injective, it has a left inverse.

  Let that inverse be $f_L : B \to A$.

Functions

- *Proof* of (4)

  We have to prove that $f_L = f_R$.

  Since $f_L \circ f = 1_A$ and $f \circ f_R = 1_B$,

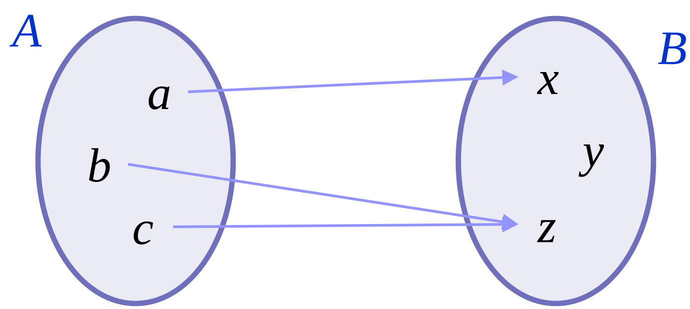
  $$f_L = f_L \circ 1_B = f_L \circ (f \circ f_R) = (f_L \circ f) \circ f_R = 1_A \circ f_R = f_R.$$

  □

Functions

- Image and inverse image:

  Let $f : A \to B$, where $A = \{a, b, c\}$, $B = \{x, y, z\}$, and
  $f = \{(a, x), (b, z), (c, z)\}$.



  - The image of the set $\{a, b\}$ under f:

    $$f = (\{a, b\}) = \{f(a), f(b)\} = \{x, z\}$$

  - The inverse image of $\{z\}$ under f is $\{b, c\}$.
  - The inverse image of $\{x, z\}$ under f is $\{a, b, c\}$.