Relations and Functions

– Part A –

Cartesian Product

- Definition:

  For sets A, B the Cartesian product, or cross product, of A and B is denoted by $A \times B$ and equals $\{(a, b) \mid a \in A, b \in B\}$.

- Ordered pair

  For $(a, b), (c, d) \in A \times B$, $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

  - *cf*. unordered pair: $\{a, b\}$
  - $(a, b) \neq \{a, b\}$, $(a, b) \neq (b, a)$

- Example:

  Let $A = \{a, b, c\}$ and $B = \{x, y\}$. Then the cross product $A \times B$ is

  $\{(u, v) \mid u \in A, v \in B\} = \{(a, x), (b, x), (c, x), (a, y), (b, y), (c, y)\}$.

Cartesian Product

- Terminologies:

  Let $A_1, A_2, \ldots, A_n$ be sets. Then the (n-fold) product of $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \times A_2 \times \cdots \times A_n$ and equals

  $$\{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i,\ 1 \le i \le n\}.$$

  The elements of $A_1 \times A_2 \times \cdots \times A_n$ are called n-tuples, although we generally use the term triple in place of 3-tuple.

  - If $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in A_1 \times A_2 \times \cdots \times A_n$, then $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ iff $a_i = b_i,\ 1 \le i \le n$.

Cartesian Product

- Note:
  - Cross product is not commutative, i.e., $A \times B \neq B \times A$.
  - Cross product is not associative, i.e., $(A \times B) \times C \neq A \times (B \times C)$.
  - $A \times \varnothing = \varnothing$
  - If $A_1 = A_2 = \cdots = A_n$, then $A_1 \times A_2 \times \cdots \times A_n = A^n$.

Cartesian Product

- Theorem: For any sets A, B, and C,
  - $A \times (B \cup C) = (A \times B) \cup (A \times C)$
  - $(B \cup C) \times A = (B \times A) \cup (C \times A)$
  - $A \times (B \cap C) = (A \times B) \cap (A \times C)$
  - $(B \cap C) \times A = (B \times A) \cap (C \times A)$.

- *Proof* : $A \times (B \cup C) = (A \times B) \cup (A \times C)$

  (i) To show $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

  Let (x, y) be an element of $A \times (B \cup C)$.

  Then, by the definition of cross product, $x \in A$ and $y \in B \cup C$.

Cartesian Product

- *Proof* :

  There are two cases:

  - CASE 1: $y \in B$

    Since $x \in A$ and $y \in B$, $(x, y) \in A \times B$.

  - CASE 2: $y \in C$

    Since $x \in A$ and $y \in C$, $(x, y) \in A \times C$.

  Since one of the two cases is true, either $(x, y) \in A \times B$ or $(x, y) \in A \times C$.

  Hence, by the definition of union of sets, $(x, y) \in (A \times B) \cup (A \times C)$.

  That is, $(x, y) \in A \times (B \cup C) \rightarrow (x, y) \in (A \times B) \cup (A \times C)$.

Cartesian Product

- *Proof* :

  Since (x, y) was an arbitrary element of $A \times (B \cup C)$, we can conclude that every elements of $A \times (B \cup C)$ is an element of $(A \times B) \cup (A \times C)$.

  Therefore, by the definition of a subset

  $$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C).$$

(ii)To show $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

. . . . .

From (i) and (ii), we conclude that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

$\square$

Cartesian Product

- *A formal Proof* : $A \times (B \cap C) = (A \times B) \cap (A \times C)$

No.	Formula	Rule	Just.	Taut.
1	$(a, b) \in A \times (B \cap C)$	AP		
2	$a \in A \wedge b \in (B \cap C)$	Def. of $\times$	1	
3	$a \in A$	T	2	I_1
4	$b \in (B \cap C)$	T	2	I_2
5	$b \in B \wedge b \in C$	Def. of $\cap$	4	
6	$b \in B$	T	5	I_1
7	$b \in C$	T	5	I_2
8	$a \in A \wedge b \in B$	T	3, 6	I_9
9	$a \in A \wedge b \in C$	T	3, 7	I_9

Cartesian Product

- *A formal Proof* : $A \times (B \cap C) = (A \times B) \cap (A \times C)$

No.	Formula	Rule	Just.	Taut.
10	$(a, b) \in A \times B$	Def. of $\times$	8	
11	$(a, b) \in A \times C$	Def. of $\times$	9	
12	$(a, b) \in A \times B \wedge (a, b) \in A \times C$	T	10, 11	I_9
13	$(a, b) \in A \times B \cap A \times C$	Def. of $\cap$	12	I_2
14	$(a, b) \in A \times (B \cap C)$	CP	1, 13	
	$\rightarrow (a, b) \in A \times B \cap A \times C$			
15	$(\forall y)\,(a, y) \in A \times (B \cap C)$	UG	14	
	$\rightarrow (a, y) \in A \times B \cap A \times C$			

Cartesian Product

- *A formal Proof* : $A \times (B \cap C) = (A \times B) \cap (A \times C)$

No.	Formula	Rule	Just.	Taut.
16	$(\forall x)(\forall y)\ (x, y) \in A \times (B \cap C)$	UG	15	
	$\rightarrow (x, y) \in A \times B \cap A \times C$			
17	$A \times (B \cap C) \subseteq A \times B \cap A \times C$	Def. of $\subseteq$	16	
18	$(a, b) \in A \times B \cap A \times C$	AP		
	. . . . .			
	$(a, b) \in A \times B \cap A \times C$	CP		
	$\rightarrow (a, b) \in A \times (B \cap C)$			
	. . . . .			
	$A \times B \cap A \times C \subseteq A \times (B \cap C)$	Def. of $\subseteq$		
	. . . . .			
	$A \times (B \cap C) = A \times B \cap A \times C$	Def. of =		

Relations

- Definitions:
  - Let A and B be sets. Then, any subset of $A \times B$ is called a relation between A and B.
  - Any subset of $A \times A$ is called a (binary) relation on A.
  - If $A_1, A_2, \ldots, A_n$ are sets then any $R \subseteq A_1 \times A_2 \times \cdots \times A_n$ is an n-ary relation on $A_1, A_2, \ldots, A_n$.
  - Let R be a binary relation between a set A and a set B. Then, the domain of R, designated by $\mathcal{D}(R)$, is

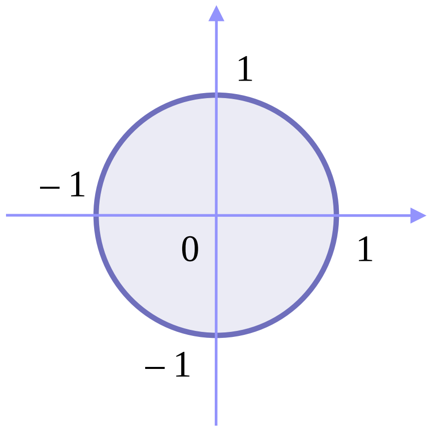
    $$\mathcal{D}(R) = \{x \mid (x, y) \in R\},$$

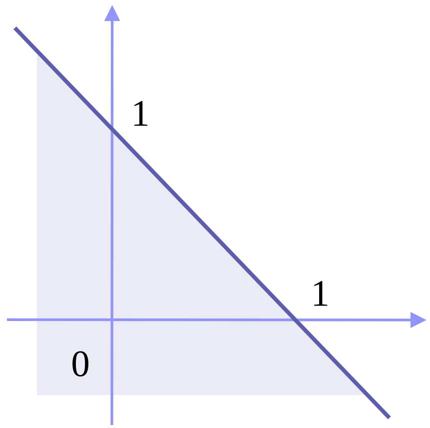
    and the range of R, designated by $\mathcal{R}(R)$, is

    $$\mathcal{R}(R) = \{y \mid (x, y) \in R\}.$$

Relations

- Examples:
  - Let $A = \{a, b, c\}$ and $B = \{x, y\}$. If $R \subseteq A \times B$ and $R = \{(b, y), (c, y)\}$, then $\mathcal{D}(R) = \{b, c\}$ and $\mathcal{R}(R) = \{y\}$.
  - If $R = \{(x, y) \mid x^2 + y^2 \leq 1\}$, then $\mathcal{D}(R) = \{x \mid -1 \leq x \leq 1\}$.
  - If $R = \{(x, y) \mid x + y < 1\}$, then $\mathcal{D}(R) = \{x \mid x \in \mathbf{R}\}$.





Relations

- **Theorem:** Let S and R be two binary relations. Then,
  - $\mathcal{D}(R \cup S) = \mathcal{D}(R) \cup \mathcal{D}(S)$
  - $\mathcal{R}(R \cup S) = \mathcal{R}(R) \cup \mathcal{R}(S)$
  - $\mathcal{D}(R \cap S) \subseteq \mathcal{D}(R) \cap \mathcal{D}(S)$
  - $\mathcal{R}(R \cap S) \subseteq \mathcal{R}(R) \cap \mathcal{R}(S)$

- A counter example for $\mathcal{D}(R) \cap \mathcal{D}(S) \subseteq \mathcal{D}(R \cap S)$:

  Let $R = \{(x, y)\}$ and $S = \{(x, z)\}$.

  Then, $\mathcal{D}(R) = \{x\}$, $\mathcal{D}(S) = \{x\}$, and $\mathcal{D}(R) \cap \mathcal{D}(S) = \{x\}$.

  But, $R \cap S = \varnothing$ and so $\mathcal{D}(R \cap S) = \varnothing$.

  Therefore, $\mathcal{D}(R) \cap \mathcal{D}(S) \not\subseteq \mathcal{D}(R \cap S)$.

Relations

- Note:
  - For $R \subseteq A \times B$, the number of relations between sets A and B is $|\wp(A \times B)| = 2^{|A \times B|} = 2^{|A| \cdot |B|}$.
  - $\varnothing$ is a relation : the smallest relation.
  - The universal relation between sets A and B is $A \times B$.

$$\overline{R} = A \times B - R.$$

- Definition:

If R is a relation between sets A and B then the converse or inverse of R, designated by R^c or R^{-1}, is given by the following:

$$R^c = \{(y, x) \mid \{(x, y) \in R\}.$$

Relations

- **Theorem:** Let R and S be two relations between the sets A and B. Then,
  - $(R^c)^c = R$
  - $(R \cup S)^c = R^c \cup S^c$
  - $(R \cap S)^c = R^c \cap S^c$
  - $(\overline{R})^c = \overline{R^c}$
  - $\mathcal{D}(R^c) = \mathcal{R}(R)$
  - $\mathcal{R}(R^c) = \mathcal{D}(R)$
  - If $R \subseteq S$ then $R^c \subseteq S^c$

Relations

- Matrix Representation of a Relation

  Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $R = \{(a, y), (b, x)\}$. Then the matrix representations of R, $\overline{R}$, and R^c are the followings:

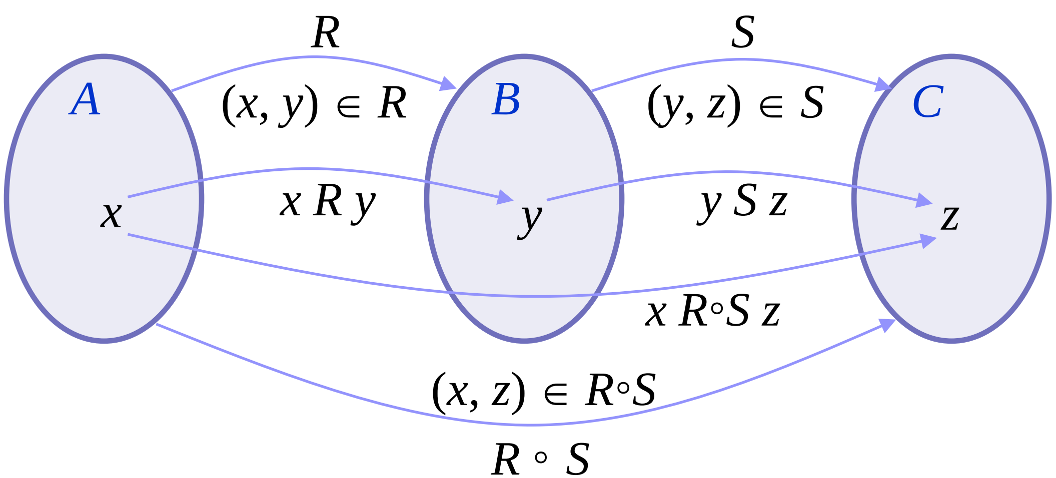
$$M_R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_{\overline{R}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad M_{R^c} = (M_R)^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Relations

- Definition:

  Let A, B, and C be sets. Let R be a relation between A and B, and S be a relation between B and C. Then the composition of R and S, denoted by $R \circ S$ or RS, is defined as follows:

$$R \circ S = \{(x, z) \mid (x, y) \in R \text{ and } (y, z) \in S\}.$$



Relations

- Caution: The composition of relations is not commutative.

$$R \circ S \neq S \circ R$$

- Definition:

  The identity relation on a set A, denoted by E_A, is defined as follows:

$$E_A = \{(a, a) \mid a \in A\}$$

- Note: For any relation R on a set A,

$$R \circ E_A = E_A \circ R = R$$

Relations

- Example:

$$A = \{a, b, c\},\ \ B = \{\alpha, \beta\},\ \ C = \{x, y, z\}$$

$$R = \{(a, \beta), (b, \alpha)\},\ \ S = \{(\beta, x), (\beta, z)\}$$

$$R \circ S = \{(a, x), (a, z)\}$$

Matrix representations:

$$M_R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad M_{E_A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{R \circ S} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = M_R \cdot M_S$$ (Note: + as $\vee$ and $\cdot$ as $\wedge$)

Relations

- **Theorem:** Let R, S and T be three relations. Then,
  - $(R \circ S) \circ T = R \circ (S \circ T)$
  - $R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$
  - $(S \cup T) \circ R = (S \circ R) \cup (T \circ R)$
  - $R \circ (S \cap T) \subseteq (R \circ S) \cap (R \circ T)$
  - $(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$

Relations

- A counter example for $(R \circ S) \cap (R \circ T) \subseteq R \circ (S \cap T)$:

  Let $R = \{(a, x), (a, y)\}$, $S = \{(x, \alpha)\}$, and $T = \{(y, \alpha)\}$.

  Then, $S \cap T = \varnothing$ and so $R \circ (S \cap T) = \varnothing$.

  But, $R \circ S = \{(a, \alpha)\}$ and $R \circ T = \{(a, \alpha)\}$.

  Therefore, $(R \circ S) \cap (R \circ T) \not\subseteq R \circ (S \cap T)$.

Relations

- Notations: If R is a relation on a set A (i.e., $R \subseteq A \times A$),
  - $R \circ R = R^2$
  - $\underbrace{R \circ R \circ \cdots \circ R}_{n} = R^n$
- Recursive definition of R^n
  - $R^0 = E$, where E is the identity relation.
  - $R^{n+1} = R \circ R^n$
- Theorem: Let R be a relation on a set A. Then,

  (i) $R^m \circ R^n = R^{m+n}$ and

  (ii) $(R^m)^n = R^{mn}$ for all $m, n \geq 1$.

- ***Proof*** of (i) $R^m \circ R^n = R^{m+n}$ (induction on m)

  (Basis step) For $m = 1$

  LHS $= R^1 \circ R^n = R \circ R^n = R^{n+1} =$ RHS (by the recursive def. of R^n)

  (Inductive step)

  Assume that $R^m \circ R^n = R^{m+n}$. (Inductive Hypothesis).

  To prove that $R^{m+1} \circ R^n = R^{m+n+1}$

  LHS $= (R \circ R^m) \circ R^n$ (by the recursive def. of R^n)

  $= R \circ (R^m \circ R^n)$ ($\because \circ$ is associative)

  $= R \circ R^{m+n}$ (by the Inductive Hypothesis)

  $= R^{m+n+1} =$ RHS (by the recursive def. of R^n)

Relations

- *Proof* of (i) $R^m \circ R^n = R^{m+n}$ (induction on m)

  We have shown that

  $$R^m \circ R^n = R^{m+n} \to R^{m+1} \circ R^n = R^{m+n+1}$$

  By this and the basis step, we conclude

  $R^m \circ R^n = R^{m+n}$ for all $m, n \geq 1$. □

- *Proof* of (ii) $(R^m)^n = R^{mn}$ (induction on n)

  (Basis step) For $n = 1$

  LHS $= (R^m)^1 = R^m = R^{m \cdot 1} =$ RHS

  (Inductive step)

  Assume that $(R^m)^n = R^{mn}$. (Inductive Hypothesis).

Relations

- *Proof* of (ii) $(R^m)^n = R^{mn}$ (induction on n)

  To prove that $(R^m)^{n+1} = R^{m(n+1)}$

  LHS $= R^m \circ (R^m)^n$ (by the recursive def. of R^n)

  $= R^m \circ R^{mn}$ (by the Inductive Hypothesis)

  $= R^{m+mn}$ (by the theorem just proved)

  $= R^{m(n+1)} =$ RHS

  □

Relations

- Theorem:

  Let R be a relation on a finite set A with cardinality n. Then, there exist s and t such that $R^s = R^t$ for $0 \le s < t \le 2^{n^2}$.

- *Proof* :

  From $R \subseteq A \times A$ we can see that the number of distinct relations on A is $|\wp(A \times A)| = 2^{|A \times A|} = 2^{n^2}$.

  Consider the sequence $R^0, R^1, R^2, \ldots, R^{2^{n^2}}$.

  There are $2^{n^2} + 1$ relations in this sequence but there are only 2^{n^2} distinct relations on A.

  Hence, there must exist s and t such that $s \neq t$ and $0 \le s < t \le 2^{n^2}$ and $R^s = R^t$. $\square$

Relations

- Theorem: Let A be a finite set with cardinality n. Let R be a relation on A such that $R^s = R^t$ with $s < t$. Let $p = t - s$. Then,

  (1) $R^{s+i} = R^{t+i}$, $i \geq 0$.

  (2) $R^{s+pk+i} = R^{s+i}$, for all k, $i \geq 0$.

  (3) If $S = \{R^0, R^1, R^2, \ldots, R^{t-1}\}$ then $R^q \in S$, for any $q \geq 0$.

- *Proof* of (1)

  (Basis step) For $i = 0$

  $R^s = R^t$ is given.

  (Inductive step)

  Assume $R^{s+i} = R^{t+i}$. (Inductive Hypothesis)

Relations

- *Proof* of (1)

  To prove that $R^{s+i+1} = R^{t+i+1}$.

  LHS $= R \circ R^{s+i}$ (by the recursive def. of R^n)

  $= R \circ R^{t+i}$ (by the Inductive Hypothesis)

  $= R^{t+i+1}$ (by the recursive def. of R^n)

  $=$ RHS

  □