Fundamentals of Logic
(Propositional Calculus)

Contents

Formal Logic

- Formal (or Symbolic) Logic

  Mathematical model of deductive thought

  - Modeling: extract important features according to purposes

    (*Ex*) The law of gravity: $F = G\frac{m\,m'}{r^2}$

    masses, distance (○) color (×)

(Q1) What are the important features of deductive thought?

  - syntax (structure), not semantics (meaning or interpretation)

    (*Ex*) Borogoves are mimsy whenever it is brillig.

    It is now brillig and this thing is a borogove.

    Therefore, this thing is mimsy.

(Q2) What does it mean for one sentence to "*follow logically*" from certain others?

(Q3) If a sentence does follow logically from certain others, what methods of proof might be necessary to establish this fact?

We need a different language since the natural language is ambiguous.

Two models are considered:

(They use different languages with different expressive power.)

- Propositional Calculus
  - Statement Calculus, Sentential Logic
  - Can express very simple, crude properties of deduction
- First-Order Predicate Calculus
  - First-Order Logic
  - Suited for mathematical deduction

Propositional Calculus

- Proposition
  - An assertion (a declarative sentence) that can take a value *true* or *false*

    (*Ex*) $x + y = 4$. This statement is false. ???

    I work hard. I am healthy. !!!

- Propositional Constants
  - stand for a particular proposition (such as W or H)

- Propositional Variables
  - stand for some propositions (such as P, Q, R, S)

Compound Propositions

- More complicated propositions resulting from combining *primitive* propositions

  (*Ex*) I work hard *and* I am healthy.

  If it rains *then* there is cloud in the sky.

Logical Connectives

Symbol	Name	Read
$\neg$	negation	not
$\wedge$	conjunction	and
$\vee$	disjunction	or
$\rightarrow$	implication (conditional)	if … then …
$\leftrightarrow$	bi-conditional	iff (if and only if)

- The meaning of logical connectives is defined by the truth table.

P	$\neg P$
F	T
T	F

P	Q	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
F	F	F	F	T	T
F	T	F	T	T	F
T	F	F	T	F	F
T	T	T	T	T	T

(Example) Jane promises that on December 26 $P \to Q$.

P: I weigh more than 120 pounds.

Q: I shall enroll in an exercise class.

$P \to Q$: If P, then Q.

Case	P	Q	$P \to Q$
1	F	F	T
2	F	T	T
3	T	F	F
4	T	T	T

(Case 4) Jane enrolls just as she said. So $P \to Q$ is true.

(Case 3) Jane breaks her promise. So $P \to Q$ is false.

(Case 2) Jane enrolls even though her weight is less than or equal to 120 pounds. However, she does not violate her promise. So $P \to Q$ is true.

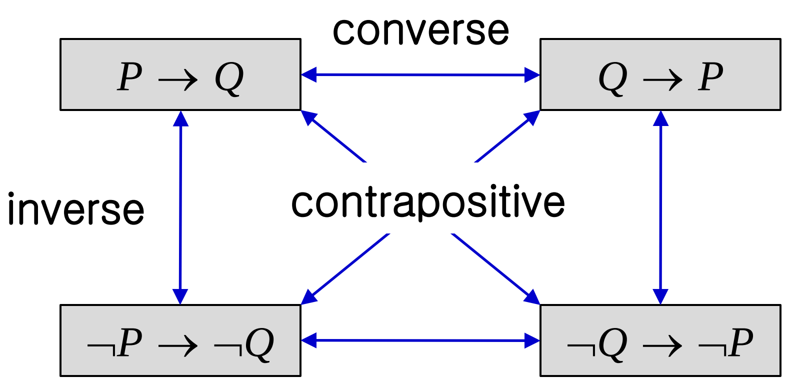
(Case 1) Jane does not violate her promise, too. So $P \to Q$ is true.

- Other Expressions for Implication

  P (antecedent) $\rightarrow Q$ (consequent)

  - If P then Q
  - P only if Q
  - P is a sufficient condition for Q
  - Q is a necessary condition for P

- Converse: $Q \rightarrow P$
- Inverse: $\neg P \rightarrow \neg Q$
- Contrapositive: $\neg Q \rightarrow \neg P$



Definition of Well-Formed Formula

A propositional well-formed formula (wff) is a grammatically correct expression, which is defined *inductively* as follows.

1. (*Basis clause*) A truth symbol (T or F), a propositional variable, or a propositional constant is a wff.
2. (*Inductive clause*) If A and B are wffs, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$ are wffs.
3. (*Extremal clause*) Only the strings formed by *finite* applications of clauses 1 and 2 are wffs.

- Precedence

  To remove parentheses, we use the following conventions.

  - Hierarchy of evaluation: (highest) $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ (lowest)
  - Operations $\wedge$, $\vee$, $\rightarrow$ are left associative.

  (*Ex*) $\mathrm{F} \wedge \neg P \vee \neg Q \rightarrow \neg Q \;\Rightarrow\; (((\mathrm{F} \wedge (\neg P)) \vee (\neg Q)) \rightarrow (\neg Q))$

  $P \rightarrow Q \rightarrow R \;\Rightarrow\; ((P \rightarrow Q) \rightarrow R)$

- Meaning (or semantics) of a wff

  The meaning of a wff is defined by its truth table.

  (*Ex*) $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$? $\qquad P \rightarrow Q \leftrightarrow \neg P \vee Q$?

$(P \to Q) \leftrightarrow (\neg P \vee Q)$?

P	Q	$P \to Q$	$\neg P \vee Q$	$(P \to Q) \leftrightarrow (\neg P \vee Q)$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	T
T	T	T	T	T

tautology

- Definition

  A tautology is a wff which takes the value true for every possible truth values assigned to the variables contained in the formula.

  (*Ex*) $\models (P \to Q) \leftrightarrow (\neg P \vee Q)$

  (Note) A tautology is true every time – not a definition!

- Definition

  A contradiction (or denial) is a wff which takes the value false for every possible truth values assigned to the variables contained in the formula.

- Definition

  A contingency is a wff which is neither a tautology nor a contradiction.

- Definition

  Let A and B be *two* wffs. The formulas A and B are said to be equivalent formulas (denoted by $A \Leftrightarrow B$), when A is true (false) iff B is true (false).

(*cf*.) Let P_1, P_2, … , P_n be the propositional variables contained in A and/or B. The formulas A and B are said to be equivalent formulas (designated by $A \Leftrightarrow B$) if they have the same value for each of the 2^n sets of truth value assignment to the propositional variables.

- Theorem 1

  A and B are equivalent formulas iff $\models A \leftrightarrow B$.

- Definition

A formula B is a substitution instance of a formula A if B is obtained from A by substituting formulas for propositional variables in A under the condition that the same formula is substituted for the same variable each time that variable appears in the formula A.

(*Ex*)

- A: $((P \to Q) \land (R \lor Q))$
- Substitute $(P \to Q)$ for P and $(R \lor S)$ for Q
- B: $(((P \to Q) \to (R \lor S)) \land (R \lor (R \lor S)))$

- Theorem 2

  A substitution instance of a tautology is a tautology.

- Theorem 3

  A formula B is equivalent to a formula A if B is obtained from A by replacing a subformula C of A with a formula D which is equivalent to C.

  (*Ex*) For $\models (P \to Q) \leftrightarrow (\neg P \vee Q)$, that is, $(P \to Q) \Leftrightarrow (\neg P \vee Q)$
  - A: $((P \to Q) \wedge (R \vee Q))$
  - B: $((\neg P \vee Q) \wedge (R \vee Q))$ by substituting $(P \to Q)$ with $(\neg P \vee Q)$
  - Then, $A \Leftrightarrow B$

- Definition

  If a formula A contains the connectives $\neg$, $\wedge$, and $\vee$ only, then the dual of A, denoted by A^{d}, is the formula obtained from A by replacing each occurrence of $\wedge$ and $\vee$ with $\vee$ and $\wedge$, respectively, and each occurrence of T and F by F and T, respectively.

  (*Ex*) $A\ : (\neg P \vee Q) \wedge (R \vee S) \wedge (\neg R \wedge \mathrm{T})$

  $A^{\mathrm{d}} : (\neg P \wedge Q) \vee (R \wedge S) \vee (\neg R \vee \mathrm{F})$

- Theorem 4: *De Morgan's Law*

  If $A(P_1, P_2, \ldots, P_{\mathrm{n}})$ is a formula and A^{d} is its dual, then

  $$\models \neg A(P_1, P_2, \ldots, P_{\mathrm{n}}) \leftrightarrow A^{\mathrm{d}}(\neg P_1, \neg P_2, \ldots, \neg P_n)$$

- Theorem 5: *Principle of Duality*

  Let A and B be two formulas and A^{d} and B^{d} be their respective duals, then

$$\models A \leftrightarrow B \text{ iff } \models A^{\mathrm{d}} \leftrightarrow B^{\mathrm{d}}$$

Conjunctive Normal Form (CNF)

A CNF is either a fundamental disjunction or a conjunction of two or more fundamental disjunctions

- A *fundamental disjunction* is either a literal or the disjunction of two or more literals
- A *literal* is a propositional variable or its negation

(*Ex*)

- P
- $P \vee \neg Q \vee R$
- $(P \vee Q) \wedge (R \vee \neg Q)$

- Conversion of wff to CNF

Every wff can be transformed to a CNF as follows.

1. Remove all occurrences of the connective $\rightarrow$ by using the equivalent formulas. $(A \rightarrow B) \Leftrightarrow \neg A \vee B$
2. Move all negations inside to create literals by using the De Morgan's law.
3. Apply distributive laws to obtain a CNF.

(*Ex*) $(P \vee Q \rightarrow R) \vee S$

$\Leftrightarrow (\neg(P \vee Q) \vee R) \vee S$; $(A \rightarrow B) \Leftrightarrow \neg A \vee B$

$\Leftrightarrow (\neg P \wedge \neg Q) \vee (R \vee S)$; De Morgan, Associative

$\Leftrightarrow (\neg P \vee R \vee S) \wedge (\neg Q \vee R \vee S)$; Distributive

Disjunctive Normal Form (DNF)

A DNF is either a fundamental conjunction or a disjunction of two or more fundamental conjunctions

- A *fundamental conjunction* is either a literal or the conjunction of two or more literals

(*Ex*)

- P
- $P \wedge \neg Q \wedge R$
- $(P \wedge Q) \vee (R \wedge \neg Q)$

Conversion of wff to DNF

Every wff can be transformed to a DNF as follows.

1. Remove all occurrences of the connective $\rightarrow$.
2. Move all negations inside.
3. Apply distributive laws.

(*Ex*) $(P \wedge Q \rightarrow R) \wedge S$

$\Leftrightarrow (\neg(P \wedge Q) \vee R) \wedge S$; $(A \rightarrow B) \Leftrightarrow \neg A \vee B$

$\Leftrightarrow (\neg P \vee \neg Q \vee R) \wedge S$; De Morgan, Associative

$\Leftrightarrow (\neg P \wedge S) \vee (\neg Q \wedge S) \vee (R \wedge S)$; Distributive

Formal Reasoning

- Formal Reasoning System (Formal Theory)

  A formal reasoning system consists of

  1. *wffs*, 2. *inference rules*, and 3. *axioms*.

- Axiom

  A wff that we wish to use as a basis of reasoning

- Inference Rule

  An inference rule maps one or more wffs, $P_1, P_2, \ldots, P_k$, called *premises* or *hypotheses* to a single wff C called the *conclusion*

$$P_1, P_2, \ldots, P_k \Rightarrow C$$

Rules of Inference

- ## Valid Consequence

Let $A_1, A_2, \ldots, A_m$, and B be formulas. Let $P_1, P_2, \ldots, P_n$ be the propositional variables occurring in $A_1, A_2, \ldots, A_m$, and B.

The formula B is said to be a *valid consequence* of the formulas A_1, $A_2, \ldots, A_m$ if for each of the 2^n sets of truth value assignments to the variables $P_1, P_2, \ldots, P_n$ the formula B takes the value T each time the formulas $A_1, A_2, \ldots, A_m$ take the value T simultaneously.

We write it

$$A_1, A_2, \ldots, A_m \models B$$

- Theorem 6

Let $A_1, A_2, \ldots, A_m$, and B be wffs. Then,

$$A_1, A_2, \ldots, A_m \models B$$

iff $A_1 \wedge A_2 \wedge \ldots \wedge A_m \models B$

iff $\models (A_1 \wedge A_2 \wedge \ldots \wedge A_m \rightarrow B)$

- Theorem 7

$$A_1, A_2, \dots, A_n \models A_i \ (1 \le i \le n)$$

- Theorem 8

If $A_1, A_2, \dots, A_n \models B_j \ (1 \le j \le p)$ and $B_1, B_2, \dots, B_p \models C$,

then $A_1, A_2, \dots, A_n \models C$.

- Inference rules are fundamental valid consequences.

$$A_1, A_2, \dots, A_n \models B$$

$$A_1, A_2, \dots, A_n \Rightarrow B$$

Derivable

The fact that formula B is derivable from formulas $A_1, A_2, \ldots, A_n$ can be demonstrated by a sequence of formulas $F_1, F_2, \ldots, F_m$ where $F_m = B$ and the presence of any formula F_i $(1 \leq i \leq m)$ is *justified* by the following two rules:

P rule (*premise* rule): $F_i = A_j$

T rule (*tautology* rule): F_i can occur in the sequence if there are formulas $F_{i1}, F_{i2}, \ldots, F_{ip}$ in the sequence before F_i and $\models (F_{i1} \wedge F_{i2} \wedge \ldots \wedge F_{ip} \rightarrow F_i)$.

We write it

$$A_1, A_2, \ldots, A_n \vdash B.$$

Proof

A proof is a finite sequence of wffs with the property that each wff in the sequence either is an axiom or can be inferred from previous wffs in the sequence.

The last wff in a proof is called a *theorem*.

(Note) Valid consequence vs. Derivable

Valid consequence	Derivable
$A_1, A_2, \ldots, A_n \models B$	$A_1, A_2, \ldots, A_n \vdash B$
Semantic (meaning)	Syntax (structure of the language)
Prove by truth table (Size 2^n grows rapidly)	Prove by rules (P rule, T rule)

(Example 1) If I get the job and work hard, then I will be promoted. If I get promoted, then I will be happy. I will not be happy. Therefore, either I will not get the job or I will not work hard.

J : I get the job. W : I work hard.

P : I get promoted. H : I will be happy.

$$J \wedge W \to P,\ P \to H,\ \neg H \ \models\ \neg J \vee \neg W$$

No.	Formula	Rule	Justification	Tautology
1	$J \wedge W \to P$	P		
2	$P \to H$	P		
3	$\neg H$	P		
4	$J \wedge W \to H$	T	1, 2	I_{13}
5	$\neg(J \wedge W)$	T	3, 4	I_{12}
6	$\neg J \vee \neg W$	T	5	E_8

(Example 2) If I study law, then I will make a lot of money. If I study archeology, then I will travel a lot. If I make a lot of money or travel a lot, then I will not be disappointed. Therefore, if I am disappointed then I did not study law and I did not study archeology.

L : I study law. M : I make a lot of money.

A : I study archeology. T : I travel a lot.

D : I am disappointed.

$$L \to M,\ A \to T,\ M \vee T \to \neg D \ \models \ D \to (\neg L \wedge \neg A)$$

No.	Formula	Rule	Justification	Tautology
1	$L \to M$	P		
2	$A \to T$	P		
3	$M \vee T \to \neg D$	P		
4	$\neg\neg D \to \neg(M \vee T)$	T	3	E_{18}
5	$D \to \neg(M \vee T)$	T	4	E_1
6	$\neg D \vee \neg(M \vee T)$	T	5	E_{16}
7	$\neg D \vee (\neg M \wedge \neg T)$	T	6	E_9
8	$(\neg D \vee \neg M) \wedge (\neg D \vee \neg T)$	T	7	E_7
9	$(\neg D \vee \neg M)$	T	8	I_1
10	$(\neg D \vee \neg T)$	T	8	I_2
11	$D \to \neg M$	T	9	E_{16}
12	$D \to \neg T$	T	10	E_{16}
13	$\neg M \to \neg L$	T	1	E_{18}
14	$\neg T \to \neg A$	T	2	E_{18}

No.	Formula	Rule	Justification	Tautology
11	$D \to \neg M$	T	9	E_{16}
12	$D \to \neg T$	T	10	E_{16}
13	$\neg M \to \neg L$	T	1	E_{18}
14	$\neg T \to \neg A$	T	2	E_{18}
15	$D \to \neg L$	T	11, 13	I_{13}
16	$D \to \neg A$	T	12, 14	I_{13}
17	$\neg D \vee \neg L$	T	15	E_{16}
18	$\neg D \vee \neg A$	T	16	E_{16}
19	$(\neg D \vee \neg L) \wedge (\neg D \vee \neg A)$	T	17, 18	I_9
20	$\neg D \vee (\neg L \wedge \neg A)$	T	19	E_7
21	$D \to \neg L \wedge \neg A$	T	20	E_{16}

(Exercise)

$$C \wedge B \to R,\ \neg B \to \neg P \models \neg R \wedge P \to \neg C$$

(Caution)

$$A \to (B \lor C),$$
$$B \to D$$
$$\models A \to (D \lor C)$$

$$A \to (B \lor C),$$
$$B \Leftrightarrow D$$
$$\models A \to (D \lor C)$$

Conditional and Indirect Proof

- (Theorem 9) Let $A_1, A_2, \ldots, A_n$ and B be wffs.

$$A_1, A_2, \ldots, A_n \models B$$

iff $A_1, A_2, \ldots, A_{n-1} \models (A_n \rightarrow B)$

iff $\models (A_1 \rightarrow (A_2 \rightarrow (\ldots (A_n \rightarrow B)) \ldots)$

- Rule of Conditional Premise (CP rule)

The formula $(F_i \rightarrow F_j)$ is justified in a derivation from the premises $A_1, A_2, \ldots, A_n$ if F_j can be derived from $A_1, A_2, \ldots, A_n, F_i$.

Additional Premise (AP)

(Ex. of CP)

$$L \to M,\ A \to T,\ M \vee T \to \neg D,\ D \ \models\ (\neg L \vee \neg A)$$

$$L \to M,\ A \to T,\ M \vee T \to \neg D \ \models\ D \to (\neg L \vee \neg A)$$

No.	Formula	Rule	Justification	Tautology
1	$L \to M$	P		
2	$A \to T$	P		
3	$M \vee T \to \neg D$	P		
4	D	**AP**		
5	$\neg\neg D$	T	4	E_1
6	$\neg(M \vee T)$	T	3, 5	I_{12}
7	$\neg M \wedge \neg T$	T	6	E_9
8	$\neg M$	T	7	I_1
9	$\neg T$	T	7	I_1
10	$\neg L$	T	1, 8	I_{12}
11	$\neg A$	T	2, 9	I_{12}
12	$\neg L \wedge \neg A$	T	10, 11	I_9
13	$D \to (\neg L \vee \neg A)$	**CP**	4, 12	

- Consistent Set

  Let $\{A_1, A_2, \dots, A_n\}$ be a set of wffs. Let $P_1, P_2, \dots, P_m$ be the propositional variables occurring in $A_1, A_2, \dots, A_n$. The set is said to be *consistent* if there exists at least one set of truth assignment to $P_1, P_2, \dots, P_m$ for which $A_1, A_2, \dots, A_n$ are simultaneously true.

- Theorem 10

  If $\{A_1, A_2, \dots, A_n\}$ is an *inconsistent* set of wffs, then $A_1 \wedge A_2 \wedge \dots \wedge A_n$ is a contradiction.

- Theorem 11

  If $A_1, A_2, \dots, A_n, B$, and C are wffs, then $A_1, A_2, \dots, A_n \models B$ iff $A_1, A_2, \dots, A_n, \neg B \models C \wedge \neg C$.

  Contradiction

- ## Proof by Contradiction

Suppose we wish to construct an indirect proof of validity of the argument $A_1 \wedge A_2 \wedge \ldots \wedge A_n \rightarrow B$. That is, we wish to prove that

$$\models (A_1 \wedge A_2 \wedge \ldots \wedge A_n \rightarrow B) \quad \text{or} \quad A_1, A_2, \ldots , A_n \models B.$$

Start the proof by writing each of the premises on a separate line with P in the rule column. Then place $\neg B$ on the next line and write "AP" in the rule column to indicate that $\neg B$ is an additional premise for indirect proof. Now treat these premises as axioms and construct a proof of a contradiction. The indirect proof gives us more information to work with because we can use both A_i and $\neg B$ as premises.

(Ex. of IP) $J \wedge W \to P,\ P \to H,\ \neg H \ \models\ \neg J \vee \neg W$

No.	Formula	Rule	Justification	Tautology
1	$J \wedge W \to P$	P		
2	$P \to H$	P		
3	$\neg H$	P		
4	$\neg(\neg J \vee \neg W)$	**AP**		
5	$\neg\neg J \wedge \neg\neg W)$	T	4	E_9
6	$J \wedge W$	T	5	E_1
7	P	T	1, 6	I_{11}
8	H	T	2, 7	I_{11}
9	$H \wedge \neg H$	T	3, 8	I_9
10	$\neg J \vee \neg W$	**IP**	4, 9	