Linear System Theory

Modelling

Modelling

■ Convolution 이 입출력관계식으로 출력을 구할 수 있지만, 임펄스 응답을 알아야 한다.

■ 임펄스 응답 : 실험적으로 구할 수 없음.

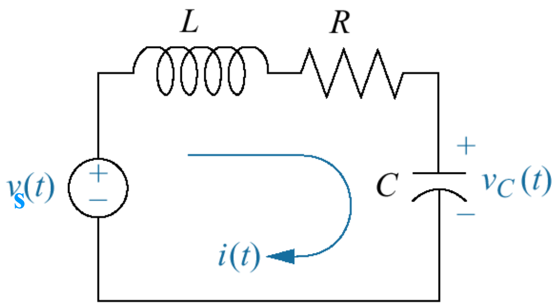
⟹ 수학적 모델링을 통한 임펄스 응답 계산

■ Modelling : 시스템에 적용되는 물리적, 전기적 특성방정식 을 이용

⟹ 일반적으로 Differential/Difference Equation 으로 표현

■ Model

(1) 시스템을 이루는 각 부분에 대한 입출력 관계



$$v_R(t) = R \cdot i(t), v_L(t) = L \cdot \frac{d}{dt} i(t), v_C(t) = \frac{1}{C}\int_0^t i(\tau) d\tau$$

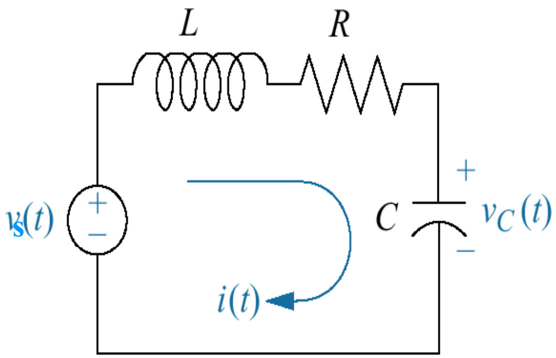
$$i(t) = dq(t)/dt, \quad q(t) = Cv_c(t)$$

(2) 전체 시스템의 입출력 관계식

Kirchhoff Voltage Law : $v_s(t) = v_R(t) + v_L(t) + v_C(t)$

■ Model

(1) 전체 시스템의 입출력 관계식 : $v_s(t) = v_R(t) + v_L(t) + v_C(t)$



$$v_R(t) = R \cdot i(t), v_L(t) = L \cdot \frac{d}{dt} i(t), v_C(t) = \frac{1}{C}\int_0^t i(\tau) d\tau$$

$$i(t) = dq(t)/dt, \quad q(t) = Cv_c(t)$$

$$\Downarrow$$

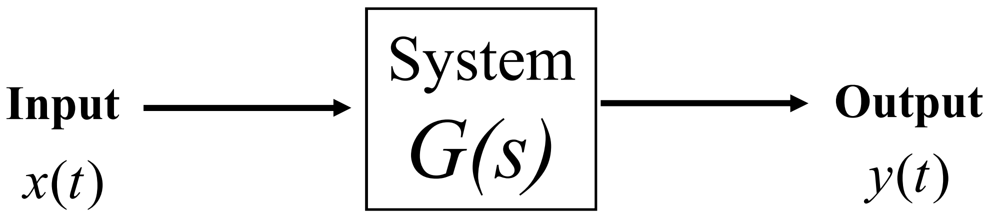
(2) **Differential Equation Model :** $LC\frac{d^2 v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$

Transfer Function

- ***Definition***. Transfer function of the system

  The ratio of the Laplace transform of the input to the Laplace transform of the output : G(s) = Y(s)/X(s)

$$G(s) = \frac{Y(s)}{X(s)} = \frac{\left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0\right)}{\left(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0\right)}$$



Laplace Transform

- Definition

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

- Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds$$

Laplace Transform(2)

- Linear Properties

$$L(kf(t)) = \int_0^\infty kf(t)e^{-st}dt = kF(s)$$

$$L(f_1(t) \pm f_2(t)) = \int_0^\infty (f_1(t) \pm f_2(t))e^{-st}dt = F_1(s) \pm F_2(s)$$

- Differentiation

$$L(\frac{d}{dt}f(t)) = \int_0^\infty [\frac{d}{dt}f(t)]e^{-st}dt = \int_0^\infty e^{-st}df(t)$$

$$= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty (\frac{d}{dt}e^{-st})f(t)dt$$

$$= f(\infty)e^{-s\infty} - f(0) + sF(s)$$

$$= sF(s) - f(0)$$

Trasfer Function Description : Motivation

■ Laplace Transform

$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$$

- Linear Constant Differential Equation
  → Algebric Relation of Rational Polynomial Function

$$LC\frac{d^2 v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

→ $$\left(LCs^2 + RCs + 1\right)V_c(s) = V(s)$$

for all zero initial condition

Transfer Function

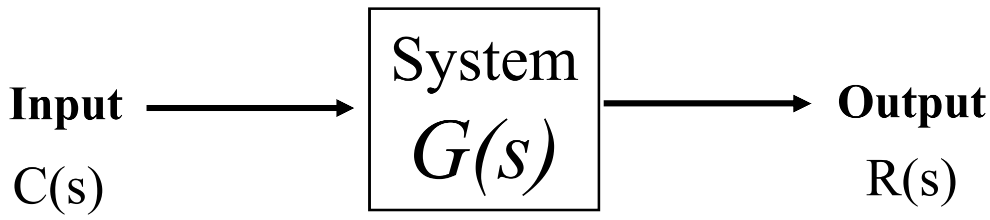
- nth order, linear, time-invariant differential equation

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t)$$

$$= b_m \frac{d^m r(t)}{dt^m} + b_{n-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

- Take the Laplace transform of both sides,

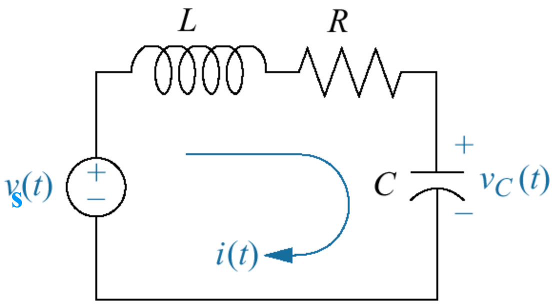
$$\left(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0\right) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0) R(s)$$



Modelling : Transfer Function

- Model

(1) Differential Equation Model



$$LC\frac{d^2v_c(t)}{dt^2}+RC\frac{dv_c(t)}{dt}+v_c(t)=v_s(t)$$

$$v_s(t)=v_R(t)+v_L(t)+v_C(t)$$

(2) Transfer Function Representation

$$\left(LCs^2+RCs+1\right)V_c(s)=V(s) \quad\Longrightarrow\quad \frac{V_c(s)}{V_s(s)}=\frac{1}{\left(LCs^2+RCs+1\right)}$$

Transfer Function : Impulse Response

- Physical Interpretation : Impulse Response

$$G(s) = Y(s) \quad \text{if} \quad X(s) = 1.$$

where $$L[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = 1$$

- The function such that R(s) = 1 is an impulse function.
- Transfer Function means the system response with respect to an impulse function.

Partial Fraction Expansion(1)

- Transfer Function

$$G(s)=\frac{Q(s)}{P(s)}=\frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_n)}$$

- Partial Fraction

$$G(s)=\frac{K_1}{s+s_1}+\frac{K_2}{s+s_2}+\cdots+\frac{K_n}{s+s_n}$$

where

$$K_i=(s+s_i)\frac{Q(s)}{P(s)}\bigg|_{s=-s_i}=\frac{Q(-s_i)}{(s_2-s_i)(s_3-s_i)\cdots(s_n-s_i)}$$