제 5 강
라플라스 변환
(Laplace tansformation)

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1. 라플라스 변환(Laplace transform)과 역변환(Inverse transform)

■ 함수 $f(t)$의 Laplace 변환공식

$$\mathcal{L}\{f(t)\}=F(s)=\int_0^{\infty} e^{-st}\cdot f(t)dt$$

$\mathcal{L}(f)=F(s)$	$\mathcal{L}^{-1}(F)=f(t)$
Laplace 변환	$F(s)$의 역변환

■ 변환의 선형성 (Linearity of Laplace transform)

$$\mathcal{L}\{af(t)+bg(t)\}=a\,\mathcal{L}(f)+b\,\mathcal{L}(g)$$

1. 라플라스 변환(Laplace transform)과 역변환(Inverse transform)

Table 1. Some functions $f(t)$ and their Laplace transform $\mathcal{L}(f)$

$f(t)$	$\mathcal{L}(f)$	$f(t)$	$\mathcal{L}(f)$
1	$\frac{1}{s}$	$\cos wt$	$\frac{s}{s^2+w^2}$
t	$\frac{1}{s^2}$	$\sin wt$	$\frac{w}{s^2+w^2}$
t^2	$\frac{2!}{s^3}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$t^n (n = 0,1,\cdots)$	$\frac{n!}{s^{n+1}}$	$\sinh at$	$\frac{a}{s^2-a^2}$
$t^a (a \text{ positive})$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$e^{at} \cdot \cos wt$	$\frac{s-a}{(s-a)^2+w^2}$
e^{at}	$\frac{1}{s-a}$	$e^{at} \cdot \sin wt$	$\frac{w}{(s-a)^2+w^2}$

1. 라플라스 변환(Laplace transform)과 역변환(Inverse transform)

■ s 영역의 s-a영역으로 이동

$$\boxed{\mathcal{L}\{e^{at}\cdot f(t)\} = F(s-a)}$$

$$\boxed{\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\cdot f(t)}$$

proof) $\mathcal{L}\{e^{at}\cdot f(t)\} = \int_0^\infty e^{-st}e^{at}\cdot f(t)dt$

$$= \int_0^\infty e^{-(s-a)t}\cdot f(t)dt = F(s-a)$$

$$F(s-a) = \int_0^\infty e^{-(s-a)t}\cdot f(t)dt = \int_0^\infty e^{-st}\{e^{at}\cdot f(t)\}dt$$

$$= \mathcal{L}\{e^{at}\cdot f(t)\}$$

1. 라플라스 변환(Laplace transform)과 역변환(Inverse transform)

예제 1

$f(t) = \cos^2 wt$, $\mathcal{L}(f)$

sol

$$f(t) = \cos^2 wt = \frac{1}{2}(1+\cos 2wt)$$

$$(\cos^2 wt) = \mathcal{L}[\frac{1}{2}(1+\cos 2wt)]$$

$$= \frac{1}{2}\mathcal{L}(1) + \frac{1}{2}\mathcal{L}(\cos 2wt)$$

$$= \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+(2w)^2}$$

$$= \frac{1}{2}(\frac{1}{s} + \frac{s}{s^2+4w^2})$$

예제 2

$f(t) = \sin 2t \cdot \cos 2t$, $\mathcal{L}(f)$

sol

$$f(t) = \sin 2t \cdot \cos 2t$$

$$= \frac{1}{2}\cdot 2\cdot \sin 2t \cdot \cos 2t$$

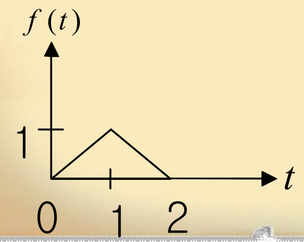
$$= \frac{1}{2}\cdot \sin 4t$$

$$\mathcal{L}(\sin 2t \cdot \cos 2t) = \frac{1}{2}\mathcal{L}(\sin 4t)$$

$$= \frac{1}{2}\cdot\frac{4}{s^2+4^2}$$

$$= \frac{2}{s^2+16}$$

■ 예제 3



$$f(t) = \begin{cases} t, & 0 \le t \le 1 \\ -t+2, & 1 \le t \le 2 \\ 0, & t \ge 2 \end{cases}$$

sol

$$\mathcal{L}(f) = \int_0^\infty e^{-st} \cdot f(t)dt$$

$$= \int_0^1 e^{-st} \cdot t\,dt + \int_1^2 e^{-st}(-t+2)\,dt + \int_2^\infty \cancelto{0}{e^{-st}} \cdot 0\,dt$$

$$= -\frac{1}{s}e^{-st}\cdot t\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st}dt - \left[-\frac{1}{s}e^{-st}\cdot t\Big|_1^2 + \frac{1}{s}\int_1^2 e^{-st}dt\right] + 2\int_1^2 e^{-st}dt$$

$$= -\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-st}\Big|_0^1 + \frac{1}{s}(2e^{-2s} - e^{-s}) - \frac{1}{s^2}e^{-st}\Big|_1^2 - \frac{2}{s}e^{-st}\Big|_1^2$$

$$= -\cancel{\frac{1}{s}e^{-s}} - \cancel{\frac{1}{s^2}e^{-s}} + \frac{1}{s^2} + \cancel{\frac{2}{s}e^{-2s}} - \cancel{\frac{1}{s}e^{-s}} - \frac{1}{s^2}e^{-2s} + \cancel{\frac{1}{s^2}e^{-s}} - \cancel{\frac{2}{s}e^{-2s}} + \cancel{\frac{2}{s}e^{-s}}$$

$$= \frac{1}{s^2} - \frac{1}{s^2}e^{-2s} = \frac{1}{s^2}(1 - e^{-2s})$$

1. 라플라스 변환(Laplace transform)과 역변환(Inverse transform)

예제 4

$$F(s)=\frac{1-7s}{(s-3)(s-1)(s+2)}\text{ , } f(t)$$

sol

$$F(s)=\frac{A}{s-3}+\frac{B}{s-1}+\frac{C}{s+2}$$

$$=\frac{A(s^2+s-2)+B(s^2-s-6)+C(s^2-4s+3)}{(s-3)(s-1)(s+2)}$$

$$=\frac{(A+B+C)s^2+(A-B-4C)s-2A-6B+3C}{(s-3)(s-1)(s+2)}$$

$$\begin{pmatrix} A+B+C=0 \\ A-B-4C=-7 \\ -2A-6B+3C=1 \end{pmatrix} \qquad A=-2,\ B=1,\ C=1$$

$$F(s)=\frac{-2}{s-3}+\frac{1}{s-1}+\frac{1}{s+2}$$

$$\mathcal{L}^{-1}(F)=f(t)=-2\cdot e^{3t}+e^{t}+e^{-2t}$$

■ 예제 5

$$F(s) = \frac{s}{(s+\frac{1}{2})^2+1}, \quad f(t)$$

sol

$$\mathcal{L}^{-1}\{\frac{s}{(s+\frac{1}{2})^2+1}\} = \mathcal{L}^{-1}\{\frac{s+\frac{1}{2}-\frac{1}{2}}{(s+\frac{1}{2})^2+1}\}$$

$$= \mathcal{L}^{-1}\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+1} - \frac{1}{2}\frac{1}{(s+\frac{1}{2})^2+1}\}$$

$$= \mathcal{L}^{-1}\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+1}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{(s+\frac{1}{2})^2+1}\}$$

$$= e^{-\frac{1}{2}t}\cdot\cos t - \frac{1}{2}e^{-\frac{1}{2}t}\cdot\sin t$$

2. 도함수의 라플라스 변환(Transforms of derivatives)

$$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0)$$

proof) $\mathcal{L}(f') = \int_0^\infty e^{-st} \cdot f'\,dt = e^{-st} \cdot f \Big|_0^\infty + s\int_0^\infty e^{-st} \cdot f\,dt$

$\boxed{f(t) = Me^{rt}}$ $\qquad = \dfrac{M}{e^{(s-r)t}}\Bigg|_0^\infty + s\,\mathcal{L}(f)$

$\boxed{f(0) = M}$ $\qquad = -f(0) + s\,\mathcal{L}(f)$

$$\begin{aligned}\mathcal{L}(f'') &= s\,\mathcal{L}(f') - f'(0)\\ &= s[s\,\mathcal{L}(f) - f(0)] - f'(0)\\ &= s^2\,\mathcal{L}(f) - s\,f(0) - f'(0)\end{aligned}$$

$$\mathcal{L}(f^n) = s^n \cdot \mathcal{L}(f) - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0) - \cdots - f^{(n-1)}(0)$$

- 예제 6

$$\mathcal{L}(t\sinh at) = \frac{2as}{(s^2-a^2)^2}$$

sol

$$f(t) = t\cdot\sinh at \qquad , \; f(0) = 0$$

$$f' = \sinh at + at\cdot\cosh at \qquad , \; f'(0) = 0$$

$$f'' = a\cosh at + a\cosh at + a^2 t\cdot\sinh at$$

$$= 2a\cosh at + a^2 t\cdot\sinh at$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s\underbrace{f(0)}_{0} - \underbrace{f'(0)}_{0}$$

$$= 2a\,\mathcal{L}(\cosh at) + a^2\mathcal{L}(t\cdot\sinh at)$$

$$= 2a\cdot\frac{s}{s^2-a^2} + a^2\mathcal{L}(f)$$

$$(s^2-a^2)\,\mathcal{L}(f) = \frac{2as}{s^2-a^2}$$

$$\boxed{\mathcal{L}(f) = \frac{2as}{(s^2-a^2)^2}}$$

- 예제 7 $y''+y=2\cos t\ ,\quad y(0)=3,\ y'(0)=4$

sol

$$\mathcal{L}(y''+y)=s^2\cdot\mathcal{L}(y)-sy(0)-y'(0)+\mathcal{L}(y)$$
$$=(s^2+1)\ \mathcal{L}(y)-3s-4$$
$$=\mathcal{L}(2\cos t)$$
$$=2\cdot\frac{s}{s^2+1}$$

$$(s^2+1)\mathcal{L}(y)=\frac{2s}{s^2+1}+3s+4$$

$$\mathcal{L}(y)=\frac{2s}{(s^2+1)^2}+\frac{3s}{s^2+1}+\frac{4}{s^2+1}$$

$$\mathcal{L}^{-1}(Y)=\mathcal{L}^{-1}\{\frac{2s}{(s^2+1)^2}\}+\mathcal{L}^{-1}(\frac{3s}{s^2+1})+\mathcal{L}^{-1}(\frac{4}{s^2+1}),\ [\ \mathcal{L}^{-1}\{F'(s)\}=-t\cdot f(t)]$$

$$\boxed{y(t)=t\cdot\sin t+3\cdot\cos t+4\cdot\sin t}$$

3. 적분함수의 라플라스 변환(Laplace transform of the integral of a function)

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\cdot F(s)}, \quad (s > 0,\ s > \gamma)$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot F(s)\right\} = \int_0^t f(\tau)d\tau}$$

proof) $g(t) = \int_0^t f(\tau)d\tau$, $\quad |f(t)| \le Me^{\gamma t}$

$$= \int_0^t Me^{\gamma\tau}d\tau = \frac{M}{\gamma}(e^{\gamma t} - 1)\ , \quad g(0) = 0$$

$$g'(t) = f(t)$$

$$\mathcal{L}(g') = \mathcal{L}(f) = s\mathcal{L}(g) - \overset{0}{g(0)}$$

$$\boxed{\mathcal{L}(g) = \frac{1}{s}\mathcal{L}(f) = \frac{1}{s}\cdot F(s)}$$

■ 예제 8

$$F(s)=\frac{4}{s^3-2s^2}, \quad f(t)$$

sol

$$F(s)=\frac{4}{s^2(s-2)}$$

$$=\frac{A}{s^2}+\frac{B}{s}+\frac{C}{s-2}$$

$$(A=-2,\ B=-1, C=1)$$

$$=-\frac{2}{s^2}-\frac{1}{s}+\frac{1}{s-2}$$

$$\boxed{f(t)=\mathcal{L}^{-1}(F)=-2t-1+e^{2t}}$$

■ 예제 9

$$F(s)=\frac{1}{s^2}\cdot\left(\frac{s-1}{s+1}\right)=\frac{s-1}{s^2(s+1)}$$

sol

$$F(s)=\frac{A}{s^2}+\frac{B}{s}+\frac{C}{s+1}$$

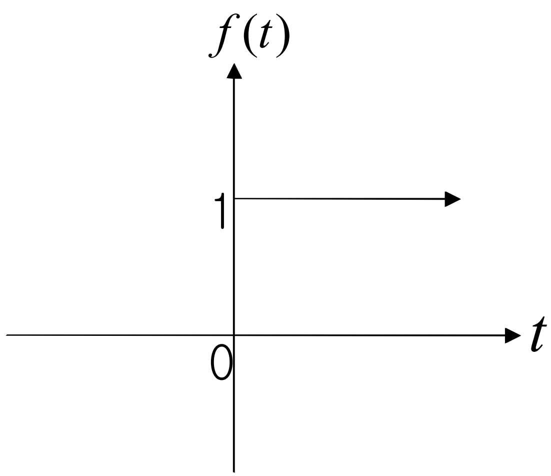
$$(A=-1,\ B=2, C=-2)$$

$$=-\frac{1}{s^2}+\frac{2}{s}-\frac{2}{s+1}$$

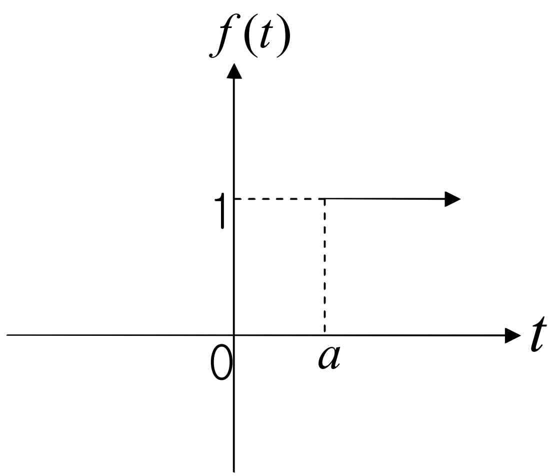
$$\boxed{\mathcal{L}^{-1}(F)=f(t)=-t+2-2e^{-t}}$$

4. 단위계단함수(Unit step function)

$$u(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1 & t \ge a \end{cases}$$



$i)\ u(t)$



$ii)\ u(t-a)$

4. 단위계단함수(Unit step function)

$$\boxed{\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}}$$

proof)

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}\cdot u(t-a)\,dt$$

$$= \cancelto{0}{\int_0^a e^{-st}\cdot 0\,dt} + \int_a^\infty e^{-st}\cdot 1\,dt$$

$$= -\frac{1}{s}e^{-st}\Big|_a^\infty = \frac{e^{-as}}{s}$$

$$f(t) = f(t-a)\cdot u(t-a) = \begin{cases} 0\ , & 0 \le t < a \\ f(t-a), & t \ge a \end{cases}$$

$$\boxed{\mathcal{L}\{f(t-a)\cdot u(t-a)\} = e^{-as}\cdot F(s)}$$

$$\boxed{\mathcal{L}\{e^{-as}\cdot F(s)\} = f(t-a)\cdot u(t-a)}$$

4. 단위계단함수(Unit step function)

■ 예제 10

$$\{f(t)\} = 4\,(t-\pi)\cdot\cos t$$

sol

$$\mathcal{L}\{4\cdot u(t-\pi)\cdot\cos t\}$$

$$= 4\cdot\frac{s}{s^2+1}\cdot e^{-\pi s}$$

$$\therefore\ \mathcal{L}(f) = \frac{4s\cdot e^{-\pi s}}{s^2+1}$$

■ 예제 11

$$\{f(t)\} = 4\,(e^{-2s} - 2\cdot e^{-5s})\,/\,s$$

sol

$$\mathcal{L}(f) = 4\cdot\frac{e^{-2s}}{s} - 8\cdot\frac{e^{-5s}}{s}$$

$$\mathcal{L}^{-1}(F) = 4\cdot u(t-2) - 8\cdot u(t-5)$$

$$\therefore\ f(t) = 4\cdot u\,(t-2) - 8\cdot u(t-5)$$

5. 변환함수의 미분(Differentiation of transforms)

$$F'(s) = \left(\int_0^\infty e^{-st} \cdot f(t)\, dt \right)'$$

$$= -\int_0^\infty e^{-st} \cdot t \cdot f(t)\, dt$$

$$= -\mathcal{L}\{t \cdot f(t)\}$$

$$\mathcal{L}\{t \cdot f(t)\} = -F'(s)$$

$$\mathcal{L}^{-1}\{F'(s)\} = -t \cdot f(t)$$

5. 변환함수의 미분(Differentiation of transforms)

$$\mathcal{L}(f) = \frac{s}{(s^2+\beta^2)^2}, \qquad f(t)$$

$$\left(\frac{\beta}{s^2+\beta^2}\right)' = \frac{-2\beta s}{(s^2+\beta^2)^2} = -2\beta \cdot \frac{s}{(s^2+\beta^2)^2}$$

$$= F'(s)$$

$$= -\frac{1}{2\beta}\mathcal{L}\{t \cdot \sin\beta t\}$$

$$\left(\frac{s}{s^2+\beta^2}\right) = \frac{1}{2\beta}\mathcal{L}\{t \cdot \sin\beta t\}$$

$$\therefore \frac{t}{2\beta} \cdot \sin\beta t = \mathcal{L}^{-1}\frac{s}{(s^2+\beta^2)^2}$$

6. 변환함수의 적분(Integration of transforms)

$$\int_s^\infty F(\tilde{s})\,d\tilde{s} = \int_s^\infty \int_0^\infty e^{-\tilde{s}t} \cdot f(t)\,dt \cdot d\tilde{s}$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-\tilde{s}t} \cdot d\tilde{s}\right]dt$$

$$= \int_0^\infty f(t)\left[-\frac{e^{-\tilde{s}t}}{t}\right]_s^\infty dt$$

$$= \int_0^\infty e^{-st} \cdot \frac{f(t)}{t}\,dt = \mathcal{L}\{\frac{f(t)}{t}\}$$

$$\boxed{\mathcal{L}\{\frac{f(t)}{t}\} = \int_s^\infty F(\tilde{s})\,d\tilde{s}}$$

$$\boxed{\mathcal{L}^{-1}\{\int_s^\infty F(\tilde{s})\,d\tilde{s}\} = \frac{f(t)}{t}}$$

- 예제 13 $\ln\left(1+\dfrac{w^2}{s^2}\right)$, $f(t)$

sol

$$-\left\{\ln\left(1+\frac{w^2}{s^2}\right)\right\}' = \frac{-s^2}{s^2+w^2}\left(\frac{s^2+w^2}{s^2}\right)'$$

$$= \frac{-s^2}{s^2+w^2}\cdot\frac{2s\cdot s^2-(s^2+w^2)2s}{s^4}$$

$$= \frac{s^2}{s^2+w^2}\cdot\frac{+2w^2 s}{s^4}$$

$$= \frac{+2w^2}{s(s^2+w^2)}$$

$$= \frac{2}{s}-2\frac{s}{s^2+w^2} = \mathcal{L}\{t\cdot f(t)\}$$

$$t\cdot f(t) = \mathcal{L}^{-1}\left(\frac{2}{s}-2\frac{s}{s^2+w^2}\right) = 2-2\cos wt$$

$$f(t) = \frac{2}{t}(1-\cos wt)$$

7. 합성(Convolution)

$$\mathcal{L}\{f(t)\} = F(s)\ ,\ \mathcal{L}\{g(t)\} = G(s)\ ,\ \mathcal{L}\{h(t)\} = H(s)\ ,$$

$$F(s)\cdot G(s) = H(s)$$

$$h(t) = (f * g)(t) = \int_0^t f(\tau)\cdot g(t-\tau)\,d\tau$$

$$\begin{aligned} proof)\ F(s)\cdot G(s) &= \int_0^\infty e^{-s\tau}\cdot f(\tau)\cdot G(s)\,d\tau \\ &= \int_0^\infty f(\tau)\int_\tau^\infty e^{-st}\cdot g(t-\tau)\,dt\,d\tau \\ &= \int_0^\infty e^{-st}\int_0^t f(\tau)\cdot g(t-\tau)\,d\tau\cdot dt \\ &= \int_0^\infty e^{-st}\cdot h(t)\,dt \\ &= \mathcal{L}\{h(t)\} \\ &= H(s) \end{aligned}$$

$$\begin{aligned} e^{-s\tau}\cdot G(s) &= \mathcal{L}\{g(t-\tau)\cdot u(t-\tau)\} \\ &= \int_0^\infty e^{-st}\cdot g(t-\tau)\cdot u(t-\tau)\,dt \\ &= \int_\tau^\infty e^{-st}\cdot g(t-\tau)\,dt \end{aligned}$$

■ 예제 14 $H(s) = \dfrac{1}{s^2(s-a)}$, $h(t)$

sol

$$i)\quad \frac{1}{s^2(s-a)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-a},\quad A = -\frac{1}{a},\ B = -\frac{1}{a^2},\ C = \frac{1}{a^2}$$

$$= -\frac{1}{a}\cdot\frac{1}{s^2} - \frac{1}{a^2}\cdot\frac{1}{s} + \frac{1}{a^2}\cdot\frac{1}{s-a}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\} = -\frac{1}{a}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{a^2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{1}{a^2}\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)$$

$$= -\frac{1}{a}\cdot t - \frac{1}{a^2} + \frac{1}{a^2}\cdot e^{at}$$

$$= \frac{1}{a^2}(e^{at} - at - 1)$$

7. 합성(Convolution)

■ 예제 14 $H(s) = \dfrac{1}{s^2(s-a)}$, $h(t)$

sol

$ii)$ $H(s) = F(s)\cdot G(s) = \dfrac{1}{s^2}\cdot\dfrac{1}{s-a}$

$$\mathcal{L}^{-1}(F) = t\ ,\quad \mathcal{L}^{-1}(G) = e^{at}$$

$$h(t) = \int_0^t \tau\cdot e^{a(t-\tau)}\,d\tau$$

$$= \int_0^t \tau\cdot e^{at}\cdot e^{a\tau}\,d\tau$$

$$= e^{at}\int_0^t \tau\cdot e^{-a\tau}\,d\tau$$

$$= e^{at}\left[-\frac{1}{a}te^{-at} - \frac{1}{a^2}e^{-at} + \frac{1}{a^2}\right]$$

$$= -\frac{1}{a}t - \frac{1}{a^2} + \frac{1}{a^2}e^{at}$$

$$= \frac{1}{a^2}(e^{at} - at - 1)$$

- 예제 15 적분방정식(Integral equation) $y(t) = t + \int_0^t y(\tau)\cdot\sin(t-\tau)\,d\tau$

sol

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{t + \int_0^t y(\tau)\cdot\sin(t-\tau)\,d\tau\}$$

$$= \mathcal{L}(t) + \mathcal{L}\{\int_0^t y(\tau)\cdot\sin(t-\tau)\,d\tau\}$$

$$Y(s) = \frac{1}{s^2} + Y(s)\cdot\mathcal{L}\{\sin t\}$$

$$= \frac{1}{s^2} + Y(s)\cdot\frac{1}{s^2+1}$$

$$(1-\frac{1}{s^2+1})Y(s) = \frac{s^2}{s^2+1} \quad Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$\boxed{\mathcal{L}^{-1}(Y) = y(t) = -t + \frac{1}{6}t^3}$$

8. Summary

- 라플라스 변환과 역변환

$$\mathcal{L}\{f(t)\}=F(s)=\int_0^{\infty} e^{-st}\cdot f(t)dt\ ,\ \mathcal{L}^{-1}(F)=f(t)$$

- 도함수의 라플라스 변환

$$\mathcal{L}(f^n)=s^n\cdot\ (f)-s^{n-1}\cdot f(0)-s^{n-2}\cdot f'(0)-\cdots-f^{(n-1)}(0)$$

- 적분함수의 라플라스 변환, $\{\int_0^t f(\tau)d\tau\}=\frac{1}{s}\cdot F(s)\ ,\ (s>0,\ s>\gamma)$
- 단위계단 함수, ***u*(*t*-*a*)**
- 변환함수의 미분, $F'(s)=-\mathcal{L}\{t\cdot f(t)\}$
- 변환함수의 적분, $\int_s^{\infty} F(\tilde{s})\,d\tilde{s}=\left\{\frac{f(t)}{t}\right\}$
- 합성, $h(t)=(f\ \ g)(t)=\int_0^t f(\tau)\cdot g(t-\tau)d\tau$