1.3 Laplace Transforms of Derivatives and Integrals

- Probably the most important property of the Laplace transformation is linearity (Theorem in the previous section). Next in order of importance comes the fact that, roughly speaking, differentiation of a f(t) corresponds simply to multiplication of the transform F(s) by S.
- <Theorem 1> (Differentiation of f(t))

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

- pf) $\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t)dt$

  ($e^{-st} \downarrow g$, $f'(t) \downarrow f'$; $\int f'g = fg - \int fg'$)

$$= (0 - f(0)) + s\int_0^\infty e^{-st} f(t)dt$$

($s\int_0^\infty e^{-st} f(t)dt = \mathcal{L}(f)$)

$$\therefore \mathcal{L}(f') = s\mathcal{L}(f) - f(0) \qquad (1)$$

* By applying (1) to the second-order derivative f'(t) We obtain

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0)$$
$$= s\{s\mathcal{L}(f) - f(0)\} - f'(0)$$
$$= s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$\therefore \mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

*Similarly

$$\therefore \mathcal{L}(f''') = s^3\mathcal{L}(f) - s^2 f(0) - sf'(0) - f''(0)$$

- <Theorem 2> (Derivative of any order)

$$\mathcal{L}\{f^{(n)}\} = s^{(n)}\mathcal{L}(f) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \cdots - f^{(n-1)}(0)$$

- Ex.1) f(t) = t^2 → $\mathcal{L}$(f) ? [Hint : $\mathcal{L}(1)=\frac{1}{s}$]

$$\text{Sol. } f'(t) = 2t \rightarrow f'(0)=0 \ \ \& \ \ f(0)=0$$

$$f''(t) = 2$$

$$\mathcal{L}\{f''(t)\} = \mathcal{L}(2)$$

$$s^2\mathcal{L}(f) - s\underbrace{f(0)}_{0} - \underbrace{f'(0)}_{0} = \frac{2}{s}$$

$$\therefore \mathcal{L}(f) = \mathcal{L}(t^2) = \frac{2}{s^3} \quad \text{: formula 3 in table 1}$$

- Ex.2) f(t) $\sin^2 t \rightarrow$ $\mathcal{L}$(f)? , f(0)=0

$$\text{sol. } f'(t) = 2\sin t\cos t = \sin 2t$$

$$\mathcal{L}(f') = \mathcal{L}(\sin 2t)$$

$$s\mathcal{L}(f') - \underbrace{f(0)}_{0} = \frac{2}{s^2+4}$$

$$\therefore \mathcal{L}(f) = \mathcal{L}(\sin^2 wt) = \frac{2}{s(s^2+4)}$$

- Ex.3) f(t) = t sinwt → $\mathcal{L}$(f)? , f(0) = 0

$$sol)\ f'(t) = \sin wt + t(\cos wt)(w) = \sin wt + wt\cos wt, \quad f'(0) = 0$$

$$f''(t) = (\cos wt)(w) + w\cos wt + wt(-\sin wt)\cdot w$$

$$= 2w\cos wt - w^2 t\sin wt = 2w\cos wt - w^2 f(t)$$

$$\mathcal{L}(f'') = 2w\,\mathcal{L}(\cos wt) - w^2\,\mathcal{L}(f)$$

$$s^2\mathcal{L}(f) - sf(0) = f'(0) \;= 2w\frac{s}{(s^2+w^2)} - w^2\,\mathcal{L}(f)$$

$$(s^2 + w^2)\,\mathcal{L}(f) = \frac{2ws}{s^2+w^2}$$

0 0

$$\therefore\ \mathcal{L}(f) = \mathcal{L}(t\sin wt) = \frac{2ws}{(s^2+w^2)^2}$$

(H.W) prove that $\mathcal{L}(t\ \cos wt) = \dfrac{s^2 - w^2}{(s^2+w^2)^2}$

- Ex.4) (A differential equation) solve the initial value problem

$$y'' + 4y' + 3y = 0 \quad , \quad y(0) = 3 \quad , \quad y'(0) = 1$$

sol) < 1st > Derive the subsidiary eqn by Laplace transformation

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}(y'') + 4\mathcal{L}(y') + 3\mathcal{L}(y) = 0$$

$$s^2Y - sy(0) - y'(0) + 4\{sY - y(0)\} + 3Y = 0$$

$$s^2Y - 3s - 1 + 4sY - 12 + 3Y$$

$$s^2Y + 4sY + 3Y = 3s + 13$$

$$\therefore Y(s) = \frac{3s+13}{s^2+4s+3} \qquad : \text{subsidiary equation}$$

< 2nd step > Solving algebrically for Y and using partial fracrions We obtain

$$Y(s) = \frac{3s+13}{s^2+4s+3} = \frac{3s+13}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3} = \frac{A(s+3)+B(s+1)}{(s+1)(s+3)}$$

□ □ $: As + 3A + Bs + B = (A+B)s + (3A+B)$

$$\therefore A + B = 3, \quad 3A + B = 13 \rightarrow 3(3-B) + B = 13$$

$$-2B = 4 \rightarrow \therefore B = -2, A = 5$$

$$\therefore Y(s) = \frac{5}{s+1} - \frac{2}{s+3}$$

< 3rd step > In oder to obtain inverse transform y(t) of Y(s), now frowTable 1, we see that

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t} \quad , \quad \mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$$

using the linearity theorem, we see that the solution of our problem is

$$\therefore\ y(t) = 5e^{-t} - 2e^{-3t}$$

As shown just above, indeed initial value problem are solved without determining a general solution.

General solution, basis : $e^{\lambda t}$.($i.e., y$ □ $e^{\lambda t}$)

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 3e^{\lambda t} = 0$$

$$(\lambda^2 + 4\lambda + 3)e^{\lambda t} = 0 \quad \rightarrow \quad *\ \lambda^2 + 4\lambda + 3 = 0$$

$$\lambda_1 = -1,\ \lambda_2 = -3$$

Basis functions of solution : $\{e^{\lambda_1 t}, e^{\lambda_2 t}\} = \{e^{-t}, e^{-3t}\}$

$\therefore$ General solution : $y(t) = Ae^{-t} + Be^{-3t} \leftarrow$ Sunbtituting the initial condition :

$$A = 5, B = 2 \quad y(0) = 3, y'(0) = 1$$

$\therefore$ Particular solution : $y(t) = 5e^{-t} + 2e^{-3t}$

(H.W) Solve the initial value problem

$$y'' + 2y' - 8y = 0 \ , y(0) = 1, y'(0) = 8$$

<Theorem 3> (Integration of f(t))

$$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}\mathcal{L}\{\mathrm{f(t)}\} \qquad (5)$$

(pf) Let $\int_0^t f(\tau)d\tau = g(t)$ (6)

$*$ Differentiate eqn(6) : $\mathrm{f(t) = g'(t)}$

$$\mathcal{L}\{\mathrm{f(t)}\} = \mathcal{L}\{\mathrm{g'(t)}\} \underset{\uparrow}{=} \mathrm{s}\mathcal{L}\{\mathrm{g(t)}\} - \mathrm{g(0)},\ \ \mathrm{g(0)} = \int_0^t f(\tau)d\tau = 0$$

From the Theorem 1

$$\therefore \mathcal{L}\{\mathrm{g(t)}\} = \frac{1}{\mathrm{s}}\mathcal{L}\{\mathrm{f(t)}\}$$

$$\therefore \mathcal{L}\int_0^t f(\tau)d\tau = \frac{1}{\mathrm{s}}\mathcal{L}\{\mathrm{f(t)}\} = \frac{1}{\mathrm{s}}\mathrm{F(s)}$$

or $\mathcal{L}\{\mathrm{f(t)}\} = \mathrm{F(s)}$

$$\therefore \mathcal{L}^{-1}\{\frac{1}{s}F(s)\} = \int_0^t f(\tau)d\tau \qquad (7)$$

- Ex.5) $\mathcal{L}(\mathrm{f}) = \dfrac{1}{\mathrm{s}^2(\mathrm{s}^2+\mathrm{w}^2)} \rightarrow f(t)\,?$

$$\text{sol.)}\quad \mathcal{L}^{-1}\left(\frac{1}{s^2+w^2}\right) = \frac{1}{w}\sin wt$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s^2+w^2}\right)\right\} = \int_0^t \frac{1}{w}\sin w\tau d\tau = \frac{1}{w}\left[\frac{1}{w}\cos w\tau\right]_0^t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\left(\frac{1}{s^2+w^2}\right)\right\} = \int_0^t \frac{1}{w^2}(1-\cos w\tau)d\tau = \frac{1}{w^2}\left[\tau - \frac{1}{w}\sin w\tau\right]_0^t$$

$$= \frac{1}{w^2(t - \frac{1}{w}\sin wt)}$$

$$\therefore\ f(t) = \frac{1}{w^2}\left(t - \frac{1}{w}\sin wt\right)$$

$$\text{(H.W)}\ \mathcal{L}(f) = \frac{1}{s^3+4s} \quad \rightarrow f(t)?$$

$$< \text{Hint} > \quad \frac{1}{s^3+4s} = \frac{1}{s(s^2+4)}$$