4

THE CONTINOUNS-TIME FOURIER TRANSFORM

4.0 Introduction

- Aperiodic signals

- Approaches to derive Fourier Transform
  - Eigenfunction approach
  - Extending Fourier series by letting the period become infinite

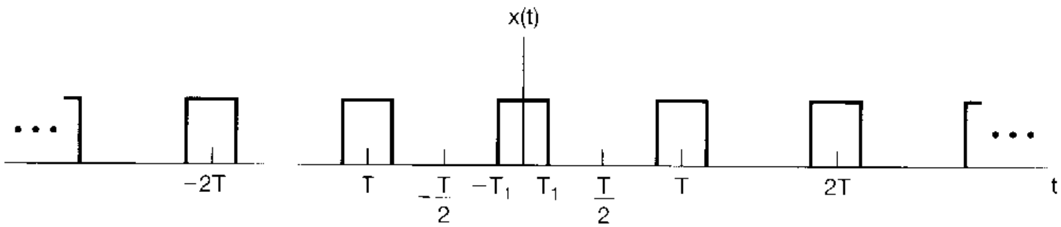
$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)e^{j\omega(t-\tau)}d\tau$$

$$y(t) = e^{j\omega t}\int_{-\infty}^{+\infty} h(\tau)e^{-j\omega\tau}d\tau$$

$$y(t) = H(j\omega)e^{j\omega t} \qquad H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)e^{-j\omega\tau}d\tau$$

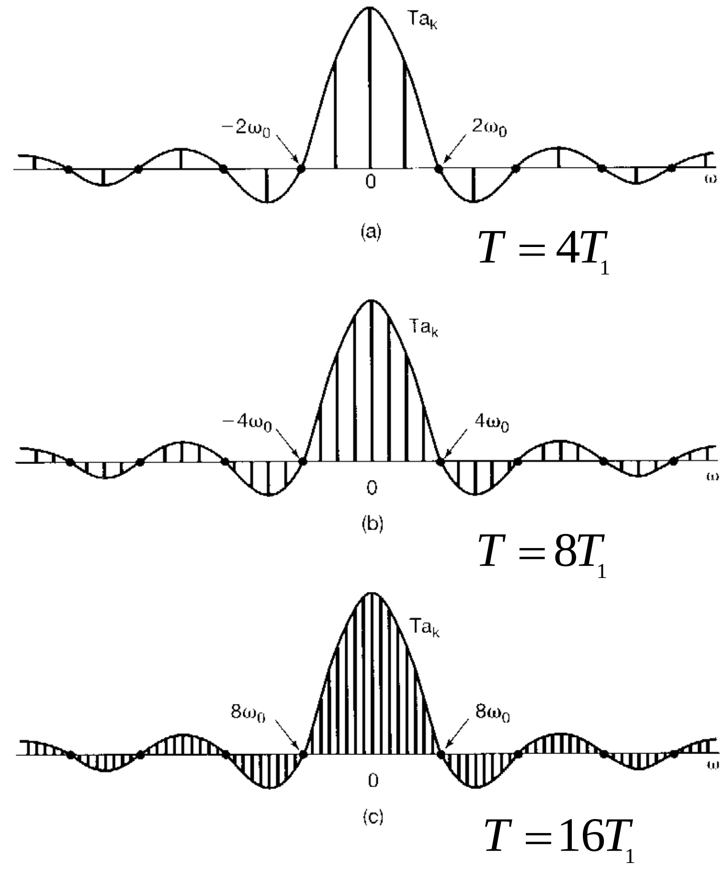
4.1 Representation of aperiodic signals
: the continuous-time Fourier transform



$$a_k = \frac{2\sin(k\omega_o T_1)}{k\omega_o T}$$

$$Ta_k = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega = k\omega_o}$$

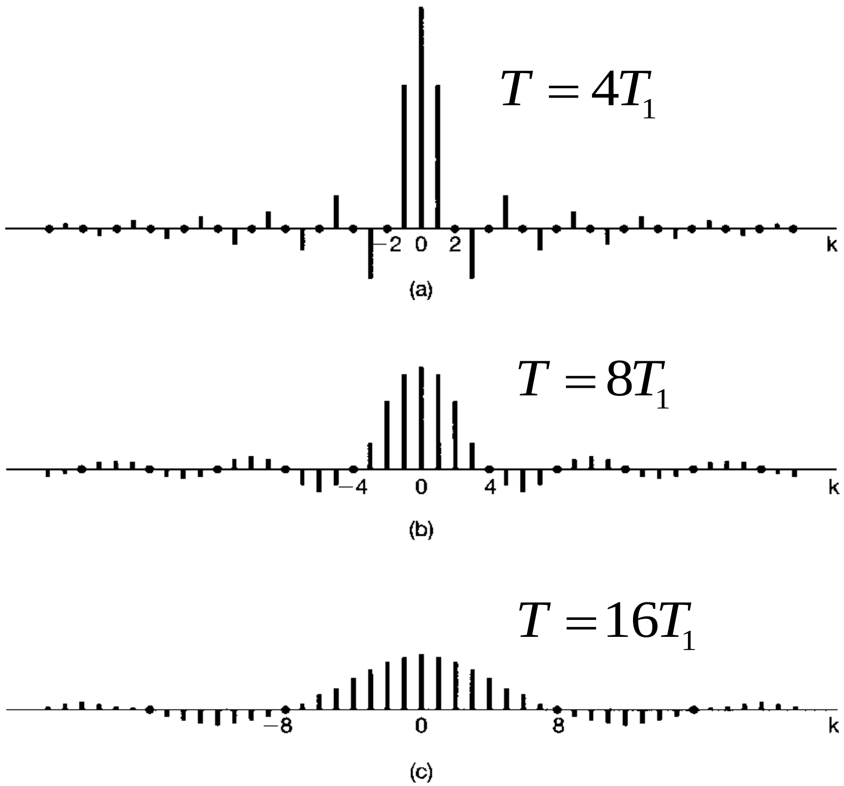
Sampled values of a specific function



4.1 Representation of aperiodic signals
: the continuous-time Fourier transform

Plots of Ta_k

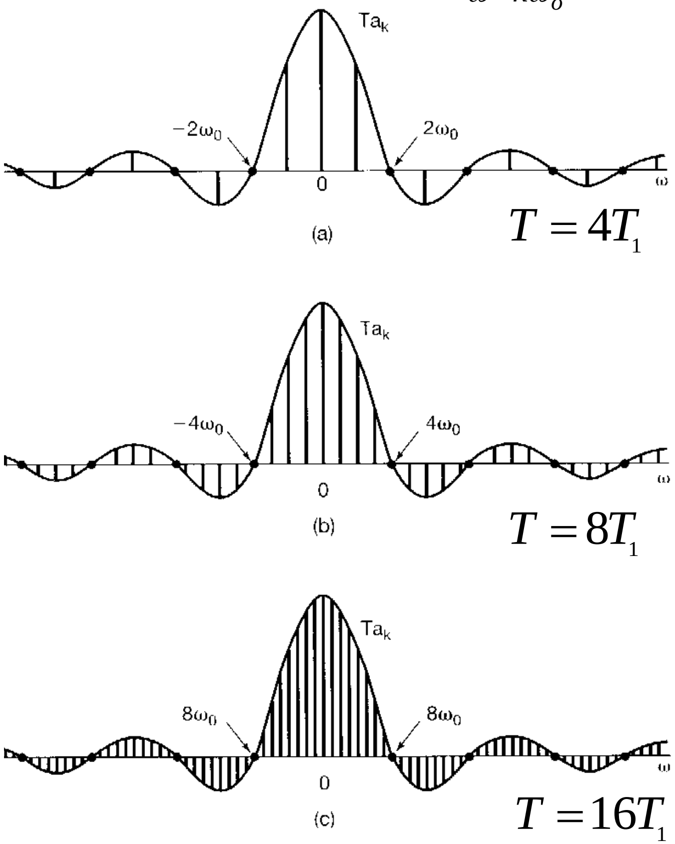
$$Ta_k = \frac{2\sin\omega T_1}{\omega}\bigg|_{\omega = k\omega_o}$$



$T = 4T_1$

$T = 8T_1$

$T = 16T_1$



$T = 4T_1$

$T = 8T_1$

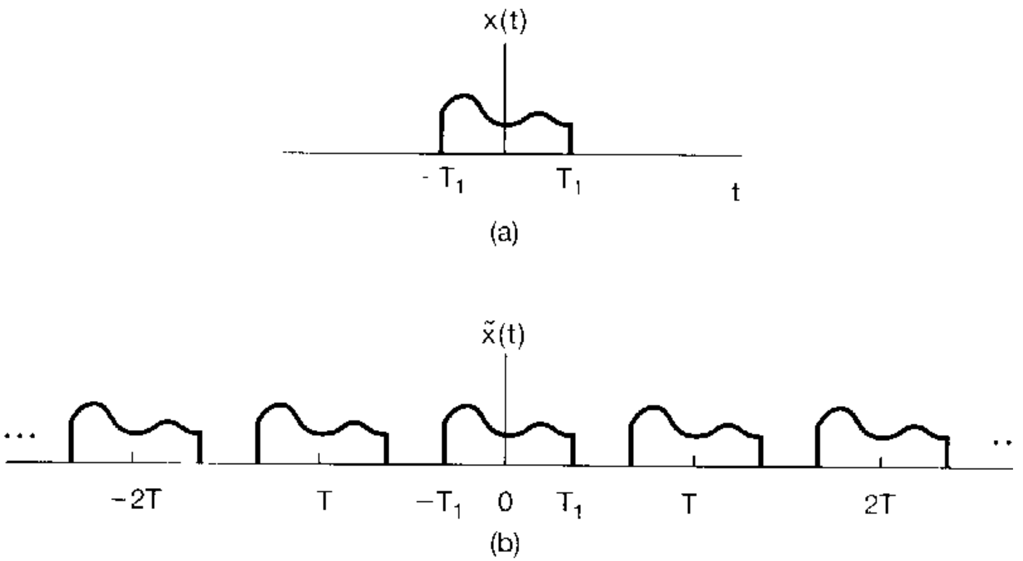
$T = 16T_1$

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_o t} dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_o t} dt$$

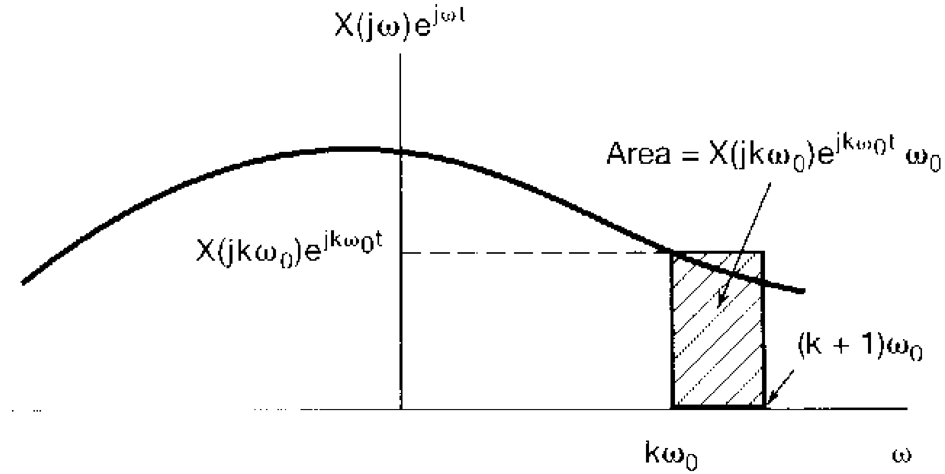
$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \Big|_{\omega = k\omega_o}$$



- Envelope of Ta_k: $X(j\omega) \triangleq \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

$$a_k = \frac{1}{T} X(jk\omega_o)$$

$$\tilde{x}(t) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(jk\omega_o)e^{jk\omega_o t} = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_o)e^{jk\omega_o t}\omega_o$$



If T goes to infinite, ω_0 becomes 0, $k\omega_0 \to \omega$, and $\tilde{x}(t) \to x(t)$.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

Inverse Fourier transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

Fourier transform
or Fourier integral of $x(t)$

4.1.2 Convergence of Fourier transform

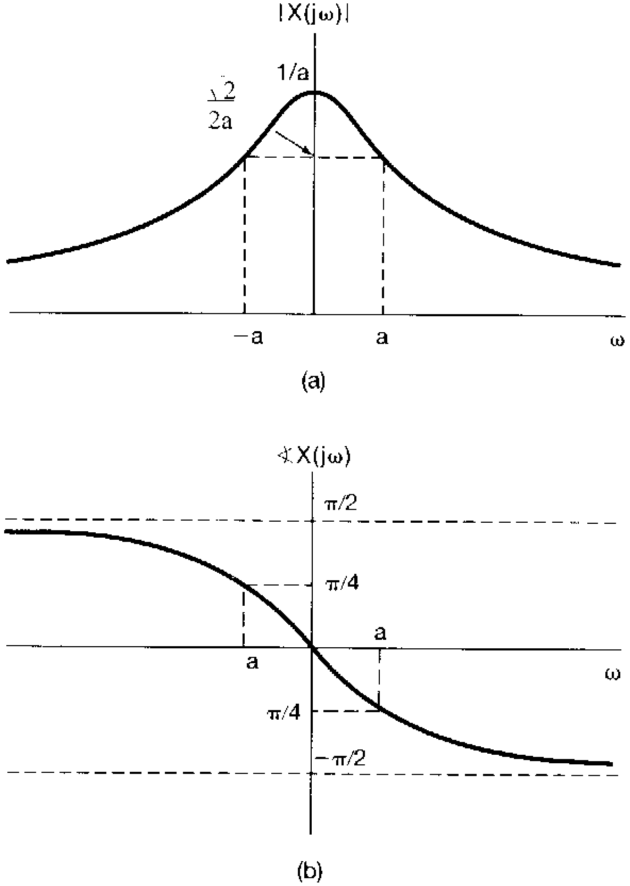
$$T \to \infty$$

Square integrable
Dirichlet conditions

Sufficient Conditions

Why not necessary conditions? Periodic signal

4.1.3 Examples of CT Fourier Transforms



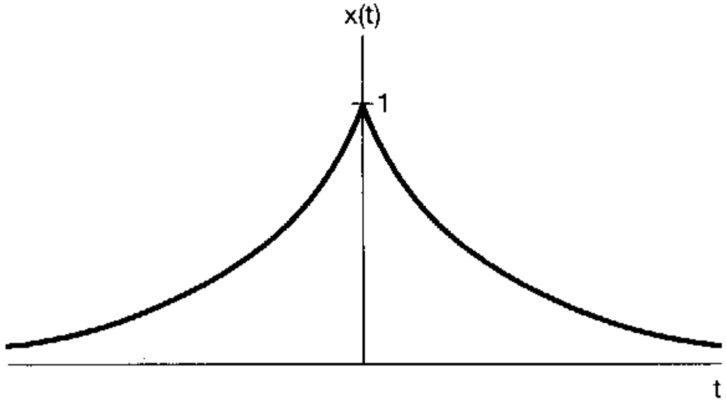
Ex. 4.1) $x(t) = e^{-at}u(t), \qquad a > 0$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

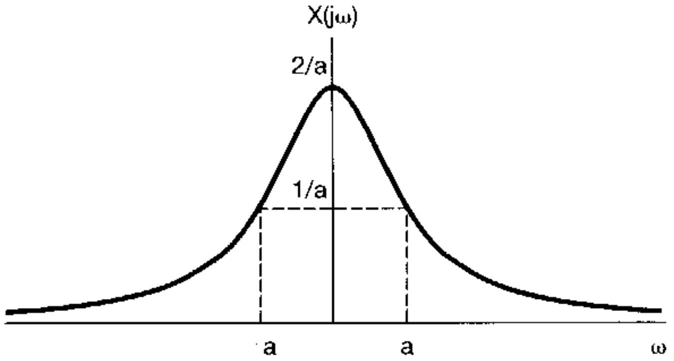
$$= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega} \qquad , a > 0$$

$$\left|X(j\omega)\right| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad , \angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Ex. 4.2) $x(t) = e^{-a|t|}, \qquad a > 0$

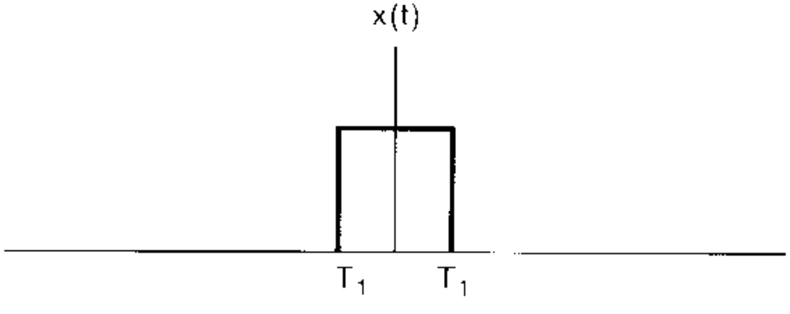


$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-a|t|} e^{-j\omega t} dt$$
$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$
$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$
$$= \frac{2a}{a^2 + \omega^2}$$



Ex. 4.3) $x(t)=\delta(t)$

$$X(j\omega)=\int_{-\infty}^{+\infty}\delta(t)e^{-j\omega t}dt=1$$



(a)

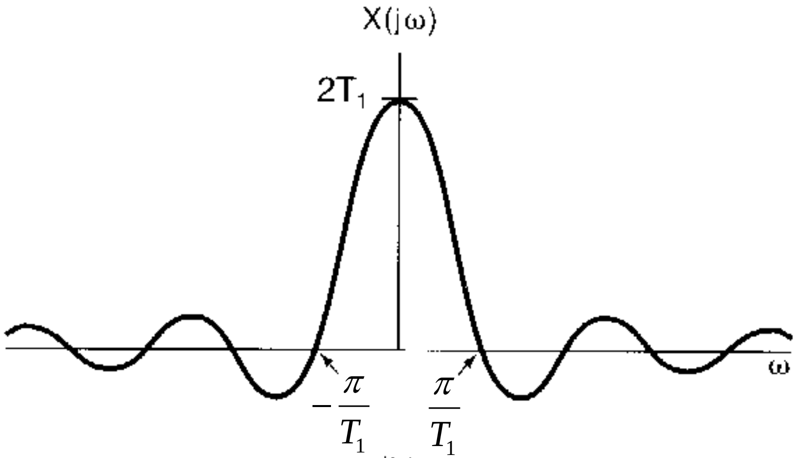
Ex. 4.4) $x(t)=\Pi\left(\frac{t}{2T_1}\right)=\begin{cases}1, & |t|<T_1\\ 0, & |t|>T_1\end{cases}$

$$X(j\omega)=\int_{-T_1}^{+T_1}e^{-j\omega t}dt$$

$$=\frac{1}{-j\omega}e^{-j\omega t}\Big|_{-T_1}^{T_1}=\frac{e^{-j\omega T_1}-e^{j\omega T_1}}{-j\omega}$$

$$=\frac{2}{\omega}\frac{e^{j\omega T_1}-e^{-j\omega T_1}}{2j}$$

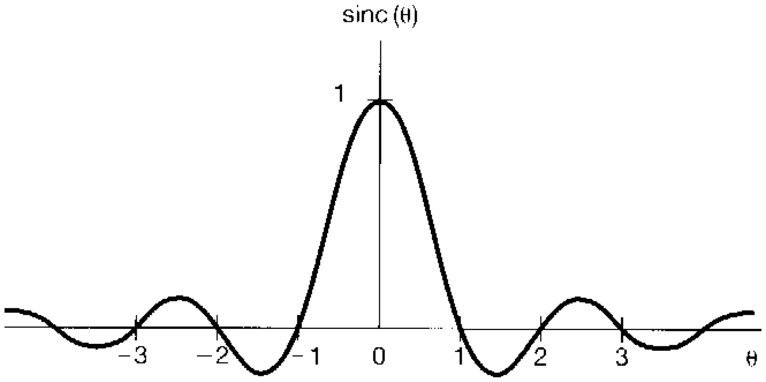
$$=2\frac{\sin\omega T_1}{\omega}\left(=2T_1\frac{\sin\pi\frac{\omega T_1}{\pi}}{\pi\frac{\omega T_1}{\pi}}=2T_1\mathrm{sinc}\left(\frac{T_1}{\pi}\omega\right)\right)$$



(b)

- Sinc function

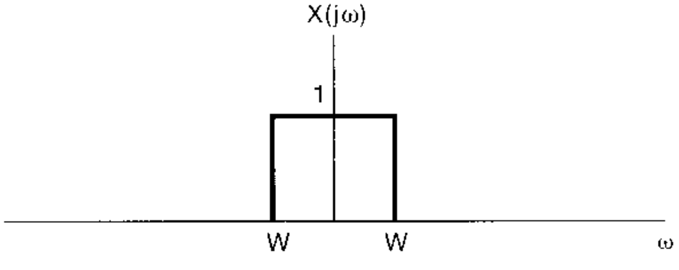
$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$$



$$\frac{\sin Wt}{\pi t} = \frac{\sin(\pi \frac{W}{\pi} t)}{\frac{\pi}{W} \cdot \pi \frac{W}{\pi} t} = \frac{W}{\pi}\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$$

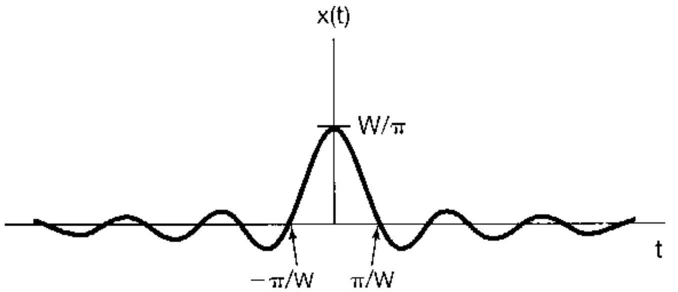
Ex. 4.5)

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$



(a)

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t} d\omega = \frac{\sin Wt}{\pi t}$$



(b)

Note) Duality $X(t) \leftrightarrow 2\pi x(-\omega)$

pf) $x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$

$$2\pi x(t) = \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$2\pi x(-t) = \int_{\omega=-\infty}^{\infty} X(j\omega)e^{-j\omega t}d\omega = \int_{\tau=-\infty}^{\infty} X(\tau)e^{-jt\tau}d\tau$$

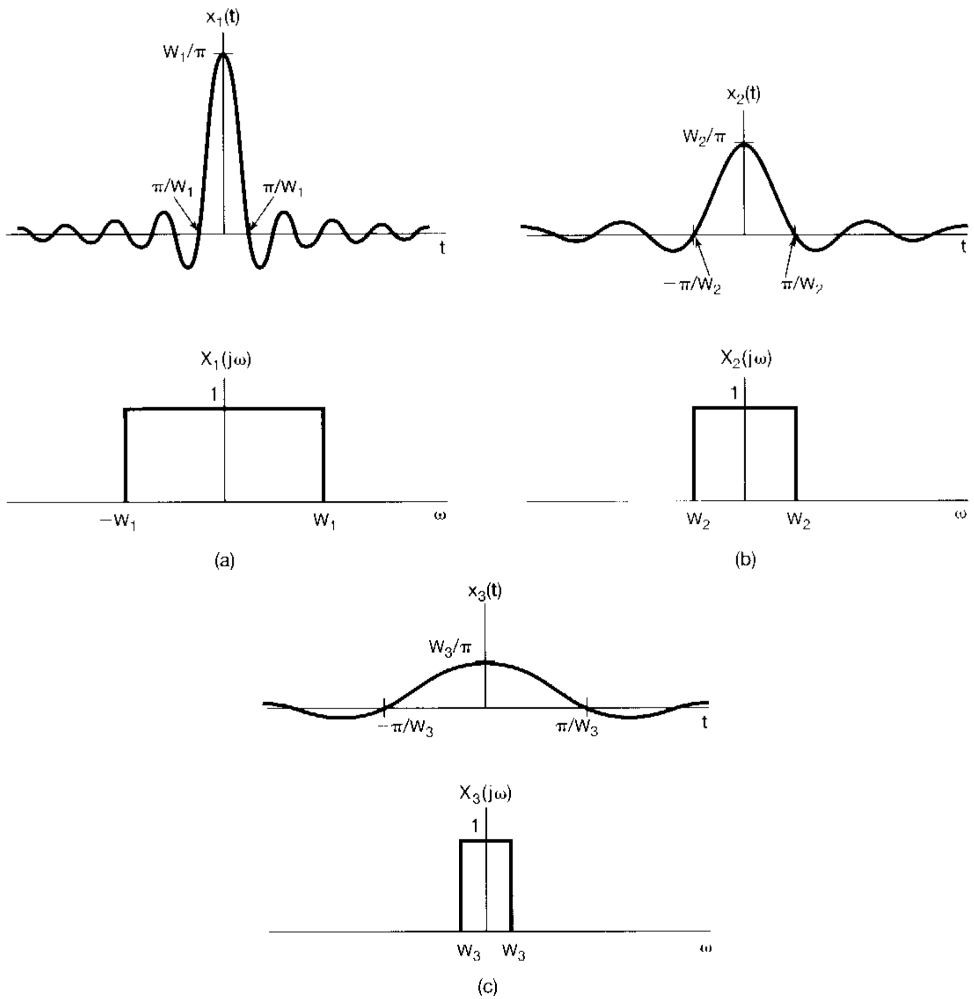
$$\therefore \quad 2\pi x(-\omega) = \int_{t=-\infty}^{\infty} X(t)e^{-j\omega t}dt$$

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \leftrightarrow X(j\omega) = 2\frac{\sin \omega T_1}{\omega} = 2T_1 \frac{\sin \pi\left(\frac{T_1}{\pi}\omega\right)}{\pi\left(\frac{T_1}{\pi}\omega\right)} = 2T_1 \mathrm{sinc}\left(\frac{T_1}{\pi}\omega\right)$$

$\omega \to t, \quad T_1 \to W_1$

$$X(t) = 2W_1 \mathrm{sinc}\left(\frac{W_1}{\pi}t\right) \leftrightarrow 2\pi x(-\omega) \Longleftrightarrow \frac{W_1}{\pi}\mathrm{sinc}\left(\frac{W_1}{\pi}t\right) = \frac{\sin W_1 t}{\pi t} \leftrightarrow x(\omega) = \begin{cases} 1, & |\omega| < W_1 \\ 0, & |\omega| > W_1 \end{cases}$$

Fourier transform pair for several different values of W



4.2 The Fourier transform for periodic signals

The Fourier transform of a periodic signal can be represented by using delta functions.

$$X(j\omega) = 2\pi\delta(\omega - \omega_o)$$

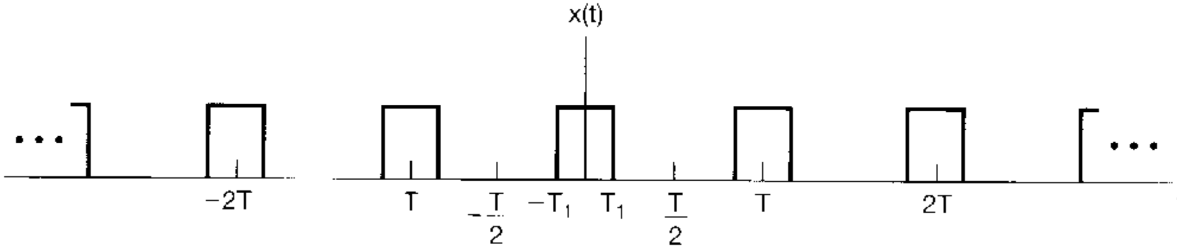
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_o) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t}$$

: Fourier series representation of a periodic signal

When $\omega_0 = 0, \quad 1 \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega)$

Ex. 4.6)

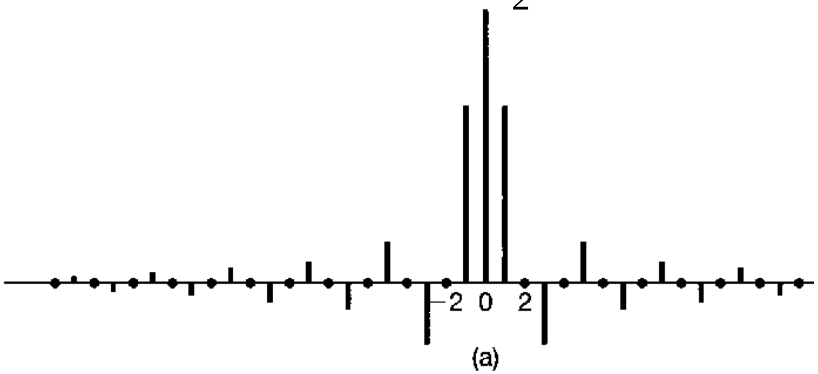


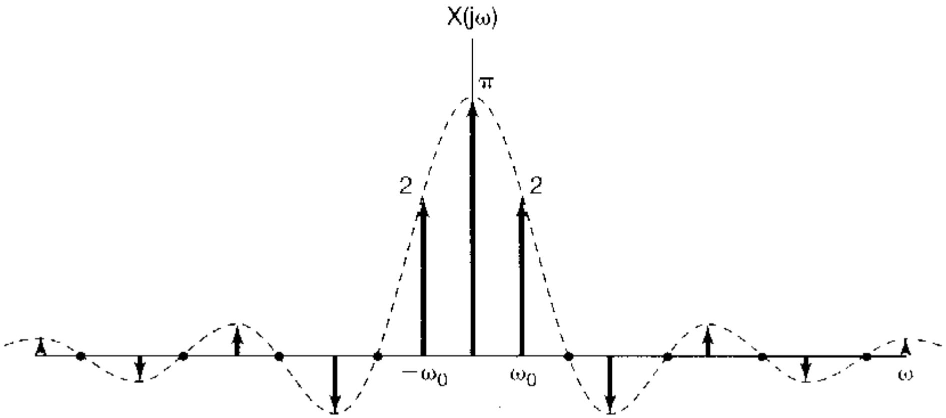
$$X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_o)$$

$$a_k = \frac{\sin(k\omega_o T_1)}{\pi k}$$

$$= \sum_{k=-\infty}^{\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$$

$$a_0 = \frac{1}{2}$$





p. 195, Fig. 3.7(a)

Ex. 4.7)

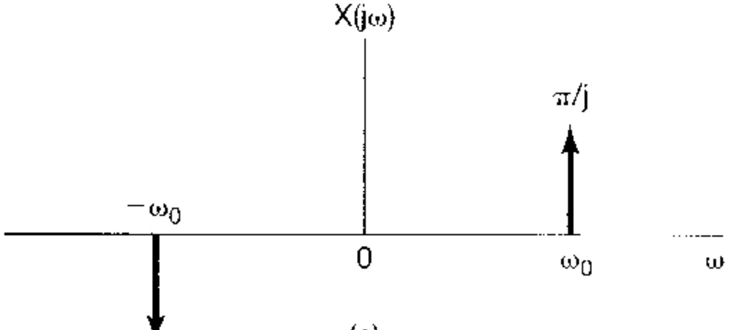
i) $x(t) = \sin \omega_0 t = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$$\therefore a_1 = \frac{1}{2j}, a_{-1} = -\frac{1}{2j}, a_k = 0 \ \ (k \neq -1,1)$$

$$\Rightarrow X(j\omega) = 2\pi a_{-1}\delta(\omega + \omega_0) + 2\pi a_1 \delta(\omega - \omega_0)$$

$$= -\frac{\pi}{j}\delta(\omega + \omega_0) + \frac{\pi}{j}\delta(\omega - \omega_0)$$

$$= j\pi\delta(\omega + \omega_0) - j\pi\delta(\omega - \omega_0)$$

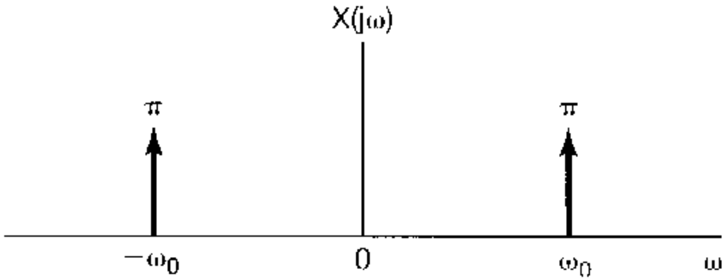


(a)

ii) $x(t) = \cos \omega_0 t$

As a similar manner

$$X(j\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$$



(b)

Ex. 4.8) $x(t)=\sum_{k=-\infty}^{+\infty}\delta(t-kT)$

$$a_k=\frac{1}{T}\int_{-T/2}^{+T/2}\delta(t)e^{-jk\omega_0 t}dt=\frac{1}{T}$$

$$X(j\omega)=2\pi\sum_{k=-\infty}^{+\infty}\left(\frac{1}{T}\right)\delta\left(\omega-k\omega_0\right)=\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$$