3.5 Properties of continuous-time Fourier series

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$ Fourier Series Pair

with Period T, Fundamental Frequency $\omega_0 = 2\pi / T$

3.5.1 Linearity

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$z(t) = Ax(t) + By(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k$$

Proof)

$$c_k = \frac{1}{T}\int_T z(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T \left[Ax(t) + By(t)\right]e^{-jk\omega_0 t}\,dt$$

$$= A\left(\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt\right) + B\left(\frac{1}{T}\int_T y(t)e^{-jk\omega_0 t}\,dt\right)$$

$$= Aa_k + Bb_k$$

3.5.2 Time shifting

$x(t-t_0)$ Time Shift

$$b_k = \frac{1}{T}\int_T x(t-t_0)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau$$

$$= e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}d\tau$$

$$= e^{-jk\omega_0 t_0}a_k = e^{-jk(2\pi/T)t_0}a_k$$

3.5.3 Time reversal

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}=\sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}, \text{ then}$$

$$x(-t)=\sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T} \qquad y(t)=x(-t)=\sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}$$

$$m=-k \qquad =\sum_{m=-\infty}^{\infty} b_m e^{jm2\pi t/T}$$

$$b_k=a_{-k}$$

$$x(t)\overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$x(-t)\overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}$$

$$x(-t)\overset{\mathcal{FS}}{\longleftrightarrow} a_k \quad \text{if } x(t) \text{ is even.} \quad \Leftrightarrow \quad a_{-k}=a_k \quad \text{(even)}$$

$$x(-t)\overset{\mathcal{FS}}{\longleftrightarrow} -a_k \quad \text{if } x(t) \text{ is odd.} \quad \Leftrightarrow \quad a_{-k}=-a_k \quad \text{(odd)}$$

3.5.4 Time scaling

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t} \qquad \text{(periodic with period } T/\alpha\text{)}$$

3.5.5 Multiplications

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l} \quad \text{convolution} \quad \text{(Prob. 3.46)}$$

3.5.6 Conjugation and conjugate symmetry (Prob. 3.42)

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$x^*(t) \overset{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}$$

If $x(t)$ is real, i.e., $x(t) = x^*(t)$, $\quad a_{-k} = a_k^* \quad$ (*conjugate symmetric*)

If $x(t)$ is real and even, then $a_k^* = a_{-k}$ and $a_k = a_{-k}$ $\quad \therefore \quad a_k = a_k^*$

If $x(t)$ is real and odd, then $a_k^* = a_{-k}$ and $a_k = -a_{-k}$ $\quad \therefore \quad a_k = -a_k^*$

3.5.7 Parseval's relation for continuous-time periodic signal

$$\frac{1}{T}\int_T |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \qquad \text{(Prob. 3.46)}$$

$$\frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2 \, dt = \frac{1}{T}\int_T |a_k|^2 \, dt = |a_k|^2$$: average power in the kth harmonic component of $x(t)$

Periodic Convolution

$$\int_T x(\tau)y(t-\tau)d\tau \xleftrightarrow{\mathcal{FS}} Ta_k b_k$$

Differentiation

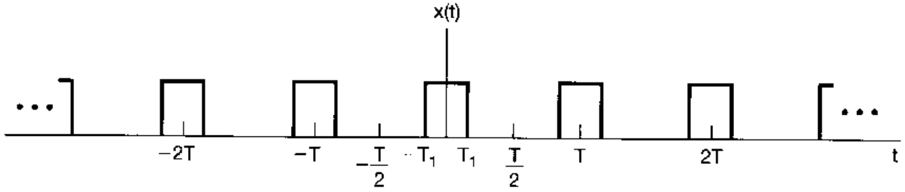
$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{FS}} jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$$

Integration

$$\int_{-\infty}^{t} x(\tau)d\tau \ \ \text{(finite valued and periodic only if } a_0 = 0\text{)}$$

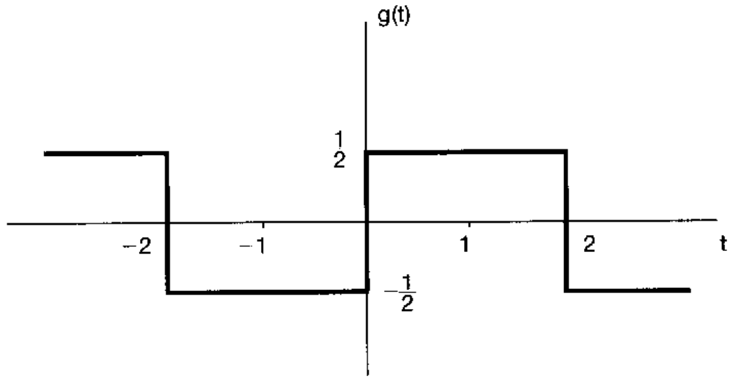
$$\xleftrightarrow{\mathcal{FS}} \left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$$

Ex. 3.6) $x(t):$



$$\Rightarrow a_k = \begin{cases} \frac{1}{2}, & k=0 \\ \frac{\sin(k\omega_0 T_1)}{k\pi}, & k \neq 0 \end{cases}$$

$$\omega_0 = \frac{2\pi}{T}, T=4, T_1=1$$



$$d_k = \frac{1}{4}\int_{-2}^{0}\left(-\frac{1}{2}\right)e^{-jk\omega_0 t}dt + \frac{1}{4}\int_{0}^{2}\frac{1}{2}e^{-jk\omega_0 t}dt$$

$$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$g(t) = x(t-1) - 1/2$$

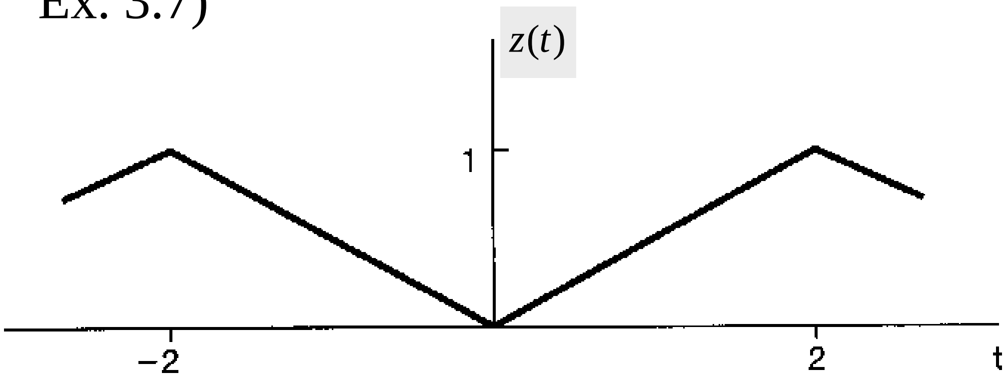
$$b_k = a_k e^{-jk\omega_0 t_0} = a_k e^{-jk\pi/2}$$

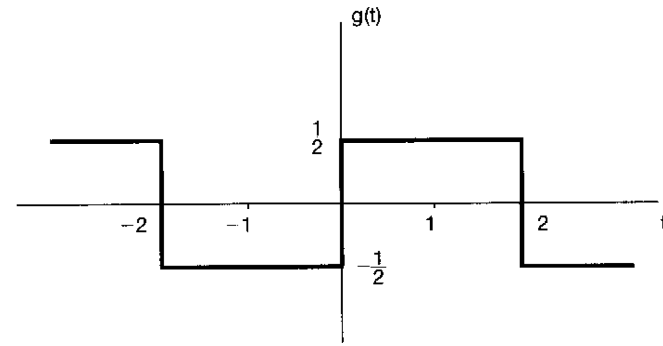
$$c_k = \begin{cases} 0, & for\ k \neq 0 \\ -\frac{1}{2}, & for\ k=0 \end{cases}$$

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & for\ k \neq 0 \\ a_0 - \frac{1}{2}, & for\ k=0 \end{cases}$$

$$d_k = \begin{cases} \frac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2} & for\ k \neq 0 \\ 0 & for\ k=0 \end{cases}$$

Ex. 3.7)





$$z(t) = \sum_{k=-\infty}^{\infty} e_k e^{jk\omega_0 t}$$

$$e_k = \frac{1}{4}\int_{-2}^{0} -\frac{t}{2} e^{-jk\omega_0 t} dt + \frac{1}{4}\int_{0}^{2} \frac{t}{2} e^{-jk\omega_0 t} dt$$

$$g(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}$$

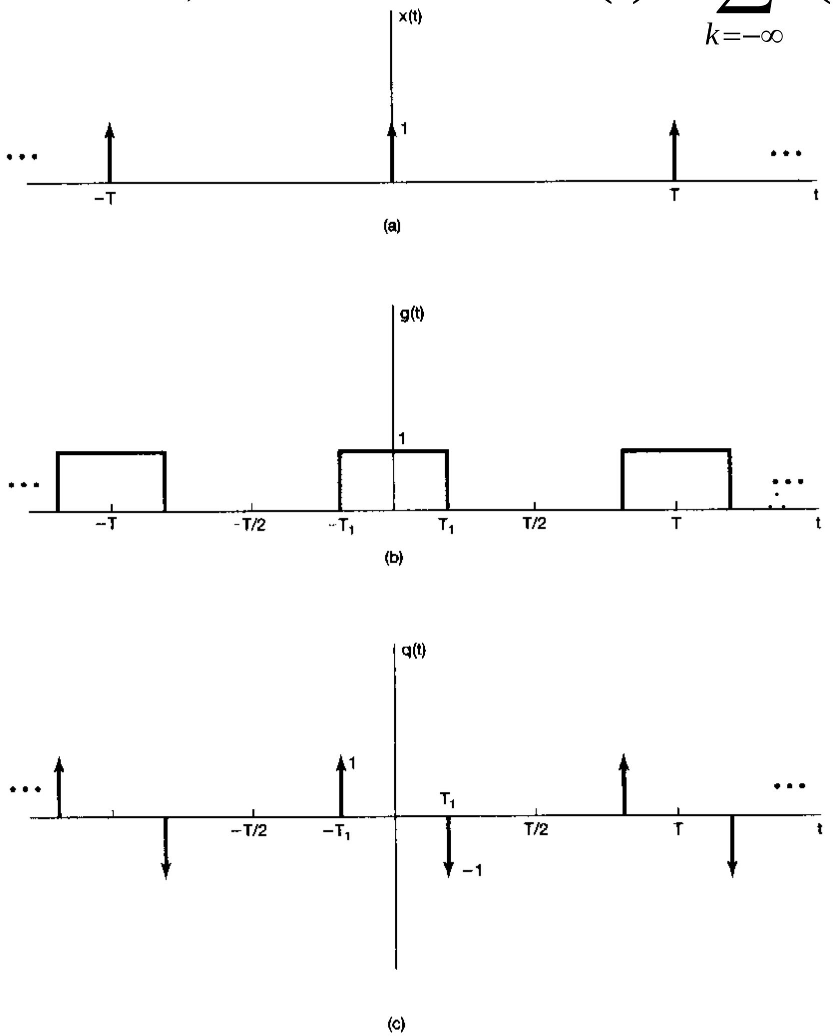
Differentiation of the current signal results in the signal of the Former example

$$g(t) = \frac{dz(t)}{dt} \Leftrightarrow d_k = jk\omega_0 e_k = jk(\pi/2)e_k$$

$$e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(k\pi/2)}{j(k\pi)^2} e^{-jk\pi/2}, \quad k \neq 0 \qquad \text{and} \quad e_0 = \frac{1}{2}$$

Ex. 3.8)

$$x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)$$



$$a_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk2\pi t/T}dt=\frac{1}{T}$$

$$x(t)=\sum_{k=-\infty}^{\infty}\frac{1}{T}e^{jk\omega_0 t}$$

$$q(t)=x(t+T_1)-x(t-T_1)$$

$$=\sum_{k=-\infty}^{\infty}b_k e^{jk\omega_0 t}$$

$$g(t)=\sum_{k=-\infty}^{\infty}c_k e^{jk\omega_0 t}$$

$$b_k=e^{jk\omega_0 T_1}a_k-e^{-jk\omega_0 T_1}a_k$$

$$b_k=jk\omega_0 c_k$$

$$\therefore\quad c_k=\frac{b_k}{jk\omega_0}$$

3.6 Fourier series representation of discrete-time periodic signals

3.6.1 Linear combinations of harmonically related complex exponentials

Discrete-time Periodic Signal

$$x[n] = x[n+N]$$

For the continuous-time periodic signals

$$x(t) = x(t+T)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

As for the discrete-time signals

$$x[n] = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 n} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/N)n}$$

Harmonically related signals

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\phi_k[n] = \phi_{k+rN}[n] \ \ (\because \phi_{k+rN}[n] = e^{j(k+rN)\omega_0 n} = e^{jk\omega_0 n} e^{jrN\omega_0 n} = e^{jk\omega_0 n} e^{jrN(2\pi/N)n} = e^{jk\omega_0 n})$$

$$\phi_N[n] = \phi_0[n]$$

$$\phi_{N+1}[n] = \phi_1[n]$$

$$x[n] = \sum_k a_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk2\pi n/N}$$

It is sufficient to include N harmonic components.

$$x[n] = \sum_{k=<N>} a_k \phi_k[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk2\pi n/N}$$

3.6.2 Determination of the Fourier series representation of a periodic signal

$$x[0] = \sum_{k=<N>} a_k e^{jk(2\pi/N)\cdot 0}$$

$$x[1] = \sum_{k=<N>} a_k e^{jk(2\pi/N)\cdot 1}$$

$$\vdots$$

$$x[N-1] = \sum_{k=<N>} a_k e^{jk(2\pi/N)\cdot(N-1)}$$

$k = < N >$
$k = 0,1,\cdots,N-1$
$k = 1,2,\cdots,N$
$k = 2,3,\cdots,N+1$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk(2\pi/N)n}$$

Multiplying both sides by $e^{-jr(2\pi/N)n}$ and summing over a period

$$\sum_{n=<N>} x[n] e^{-jr(2\pi/N)n} = \sum_{n=<N>} \sum_{k=<N>} a_k e^{jk(2\pi/N)n} e^{-jr(2\pi/N)n} = \sum_{n=<N>} \sum_{k=<N>} a_k e^{j(k-r)(2\pi/N)n}$$

$$= \sum_{k=<N>} a_k \underbrace{\sum_{n=<N>} e^{j(k-r)(2\pi/N)n}}_{?}$$

$$\sum_{n=<N>} e^{j(k-r)(2\pi/N)n} = \begin{cases} N, & k-r = 0, \pm N, \pm 2N, \ldots \\ 0, & otherwise \end{cases}$$

Problem 3.54 Show that

$$\sum_{n=<N>} e^{j(k-r)(2\pi/N)n} = \begin{Bmatrix} N, & k-r=0, \pm N, \pm 2N, \ldots \\ 0, & otherwise \end{Bmatrix}$$

i) $k-r \neq mN$ for $m=0, \pm 1, \pm 2,$ etc.

$$\sum_{n=0}^{N-1} \exp\left[j(k-r)\tfrac{2\pi}{N}n\right] = \frac{1-\exp\left[j(k-r)\tfrac{2\pi}{N}N\right]}{1-\exp\left[j(k-r)\tfrac{2\pi}{N}\right]} = 0$$

ii) $k-r = mN$

$$\sum_{n=0}^{N-1} \exp\left[j(k-r)\tfrac{2\pi}{N}n\right] = \sum_{n=0}^{N-1} \exp\left[jmN \cdot \tfrac{2\pi}{N} \cdot n\right]$$

$$= \sum_{n=0}^{N-1} \exp\left[j2\pi \cdot mn\right] = N$$

$$\therefore \quad \sum_{n=0}^{N-1} \exp\left[j(k-r)\tfrac{2\pi}{N}n\right] = N\delta(k-r-mN)$$

Discrete Fourier series pair

$$x[n] = \sum_{k=<N>} a_k e^{jkw_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

synthesis equation

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n} = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

analysis equation

$\{a_k\}$ Fourier series coefficients of $x[n]$

or

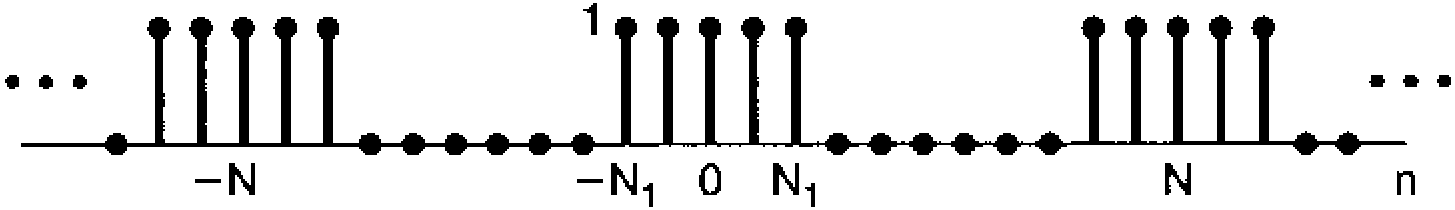
Spectral coefficients of $x[n]$

$$x[n] = a_0\phi_0[n] + a_1\phi_1[n] + \dots + a_{N-1}\phi_{N-1}[n]$$

$$x[n] = a_1\phi_1[n] + a_2\phi_2[n] + \dots + a_N\phi_N[n]$$

$a_k = a_{k+N}$ ⇨ a_k is also a periodic sequence

Ex. 3.12)



$$a_k = \frac{1}{N}\sum_{n=-N_1}^{+N_1} e^{-jk(2\pi/N)n}$$

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)}$$

$$= \frac{1}{N} e^{-jk(2\pi/N)N_1}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}$$

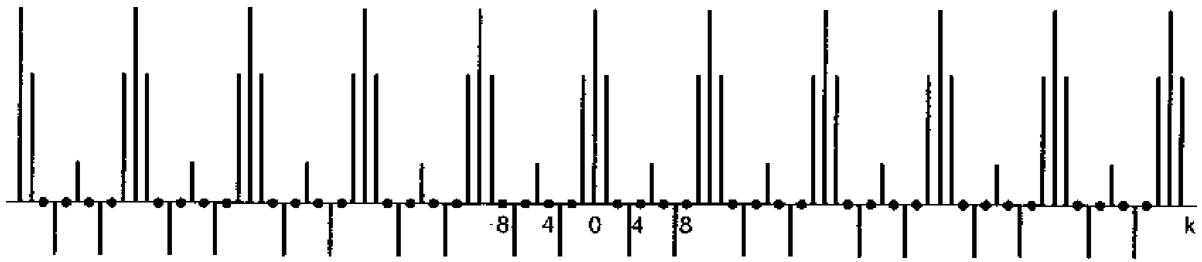
$$a_k = \frac{1}{N} e^{jk(2\pi/N)N_1}\left(\frac{1-e^{-jk2\pi(2N_1+1)/N}}{1-e^{-jk2\pi/N}}\right)$$

$$= \frac{1}{N}\frac{e^{-jk(2\pi/2N)}}{e^{-jk(2\pi/2N)}}\left(\frac{e^{jk2\pi(N_1+1/2)/N}-e^{-jk2\pi(N_1+1/2)/N}}{e^{jk(2\pi/2N)}-e^{-jk(2\pi/2N)}}\right)$$

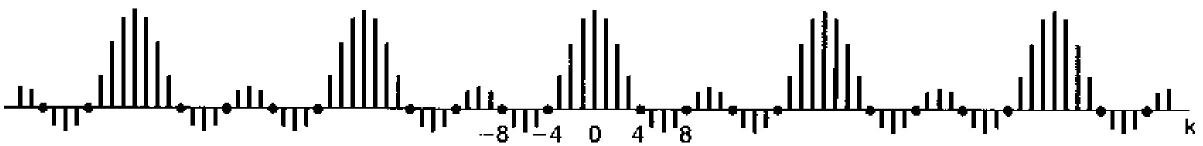
$$= \frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)} \qquad k \neq 0, \pm N, \pm 2N, \ldots$$

$$a_k = \frac{2N_1+1}{N}, \qquad k = 0, \pm N, \pm 2N, \ldots$$

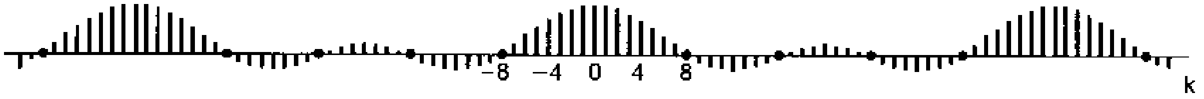
Plots of Na_k



(a) $N = 10$

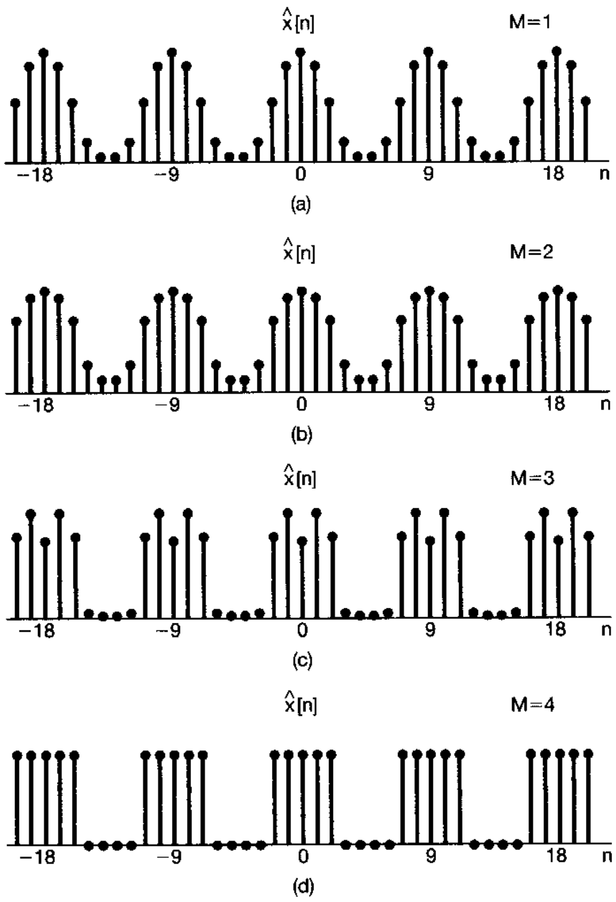


(b) $N = 20$



(c) $N = 40$

Partial sums for the periodic square wave



$$\hat{x}[n] = \sum_{k=-M}^{+M} a_k e^{jk(2\pi/N)n} \qquad N : \text{odd}$$

$$\hat{x}[n] = \sum_{k=-M+1}^{+M} a_k e^{jk(2\pi/N)n} \qquad N : \text{even}$$

$$x[n] \underset{\text{Recover}}{\overset{\mathcal{FS}}{\rightleftarrows}} a_k : \text{equivalent set}$$

Any discrete-time periodic seq. $x[n]$ is completely specified by a finite number N of parameters

3.7 Properties of discrete-time Fourier series

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$

3.7.1 Multiplication

$$x[n]y[n] \xleftrightarrow{\mathcal{FS}} d_k = \sum_{l=<N>} a_l b_{k-l}$$: periodic convolution

3.7.2 First difference

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

$$b_k = \frac{1}{N} \sum_{n=<N>} x[n-1] e^{-jk(2\pi/N)n} = \frac{1}{N} e^{-jk(2\pi/N)} \sum_{n=<N>} x[n-1] e^{-jk(2\pi/N)(n-1)}$$

$$= e^{-jk(2\pi/N)} a_k$$

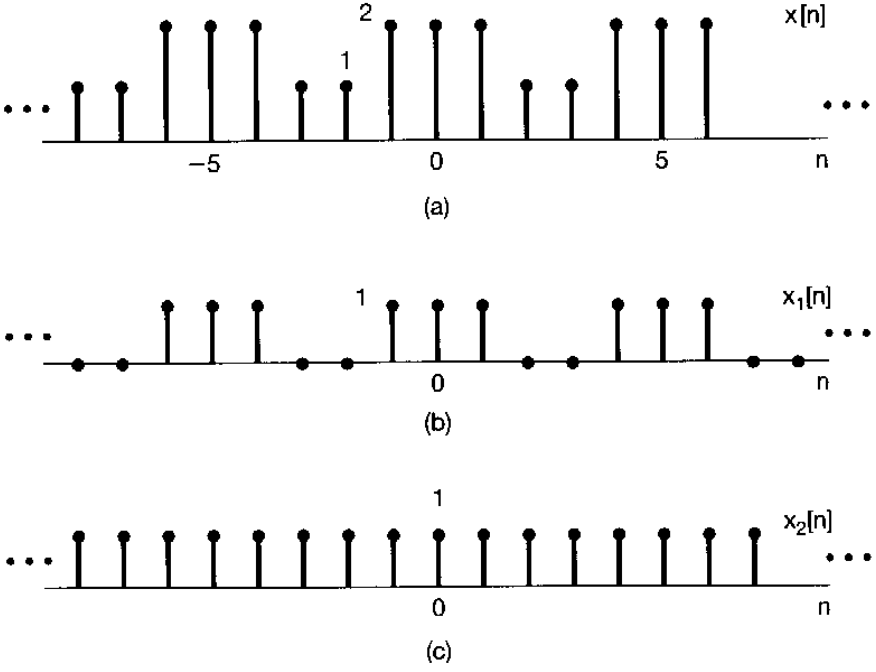
$$x[n] - x[n-1] \xleftrightarrow{\mathcal{FS}} a_k (1 - e^{-jk(2\pi/N)})$$

3.7.3 Parseval's relation to discrete-time periodic signals

$$\frac{1}{N}\sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$$

3.7.4 Examples

Ex. 3.14)



$$x[n] = x_1[n] + x_2[n]$$

$$a_k = b_k + c_k$$

$$b_k = \begin{cases} \dfrac{\sin(3k\pi/5)}{5\sin(k\pi/5)} & \text{for } k \neq 0, \pm 5, \pm 10, \ldots \\ \dfrac{3}{5} & \text{for } k = 0, \pm 5, \pm 10, \ldots \end{cases}$$

$$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1$$

Example 3.14

1. $x[n]$ is periodic with period $N = 6$.

2. $\sum_{n=0}^{5} x[n] = 2.$ ⇒ $a_0 = \frac{1}{3}$

3. $\sum_{n=2}^{7} (-1)^n x[n] = 1.$

$$(-1)^n = e^{-j\pi n} = e^{-j(2\pi/6)3n} \Rightarrow a_3 = \frac{1}{6}$$

4. $x[n]$ has the minimum power per period. ⇒ $a_1 = a_2 = a_4 = a_5 = 0$

$$\therefore \quad x[n] = a_0 + a_3 e^{j\pi n}$$

$$= \frac{1}{3} + \frac{1}{6}(-1)^n$$

3.8 Fourier series and LTI systems

$$H(s)=\int_{-\infty}^{\infty}h(\tau)e^{-s\tau}d\tau \qquad H(s),H(z):\text{system functions}$$

$$H(z)=\sum_{k=-\infty}^{\infty}h[k]z^{-k}$$

$$H(j\omega)=\int_{-\infty}^{\infty}h(t)e^{-jwt}dt \qquad H(j\omega):\text{frequency response}$$

$e^{j\omega n}$: input $\qquad h[n]$: impulse response of the LTI system

$$y[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]=\sum_{k=-\infty}^{\infty}h[k]e^{j\omega(n-k)}$$
$$=e^{j\omega n}H(e^{j\omega})$$

$$H(e^{j\omega})=\sum_{k=-\infty}^{\infty}h[k]e^{-j\omega k} \qquad :\text{frequency response}$$

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Rightarrow y(t)=\sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

Ex. 3.16)

$$x(t)=\sum_{k=-3}^{+3} a_k e^{jk2\pi t} \quad (a_0=1, a_1=a_{-1}=\frac{1}{4}, a_2=a_{-2}=\frac{1}{2}, a_3=a_{-3}=\frac{1}{3})$$

$$h(t)=e^{-t}u(t)$$

$$H(j\omega)=\int_0^{\infty} e^{-\tau} e^{-j\omega\tau} d\tau = -\frac{1}{1+j\omega} e^{-\tau} e^{-j\omega\tau}\Big|_0^{\infty} = \frac{1}{1+j\omega}$$

$$y(t)=\sum_{k=-3}^{3} b_k e^{jk2\pi t}, \quad \text{with } b_k = a_k H(jk2\pi)$$

$$b_0 = 1$$

$$b_1 = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right), \quad b_{-1} = \frac{1}{4}\left(\frac{1}{1-j2\pi}\right)$$

$$b_2 = \frac{1}{2}\left(\frac{1}{1+j4\pi}\right), \quad b_{-2} = \frac{1}{2}\left(\frac{1}{1-j4\pi}\right)$$

$$b_3 = \frac{1}{3}\left(\frac{1}{1+j6\pi}\right), \quad b_{-3} = \frac{1}{3}\left(\frac{1}{1-j6\pi}\right)$$

Since $b_k^* = b_{-k}$, $\quad y(t) = b_0 + 2\sum_{k=1}^{3} \mathrm{Re}\{b_k e^{j2\pi kt}\}$

$$y(t) = 1 + 2\sum_{k=1}^{3} D_k \cos(2\pi kt + \theta_k) \quad or \quad y(t) = 1 + 2\sum_{k=1}^{3} \left[E_k \cos 2\pi kt - F_k \sin 2\pi kt\right],$$

where $b_k = D_k e^{j\theta_k} = E_k + jF_k$

ex) $\quad D_1 = |b_1| = \dfrac{1}{4\sqrt{1+4\pi^2}}, \quad \theta_1 = \angle b_1 = -\tan^{-1}(2\pi)$

$$E_1 = \mathcal{Re}\{b_1\} = \frac{1}{4(1+4\pi^2)}, \quad F_1 = \mathcal{Im}\{b_1\} = -\frac{\pi}{2(1+4\pi^2)}$$

Ex. 3.17) Discrete-time system

$$y[n]=\sum_{k=<N>} a_k H(e^{j(2\pi k/N)})e^{jk(2\pi/N)n}$$

$$h[n]=\alpha^n u[n]$$

$$x[n]=\cos\left(\frac{2\pi n}{N}\right)=\frac{1}{2}e^{j(2\pi/N)n}+\frac{1}{2}e^{-j(2\pi/N)n}$$

$$H(e^{j\omega})=\sum_{n=0}^{\infty}\alpha^n e^{-j\omega n}=\sum_{n=0}^{\infty}(\alpha e^{-j\omega})^n$$

$$H(e^{j\omega})=\frac{1}{1-\alpha e^{-j\omega}}$$

$$y[n]=\frac{1}{2}H(e^{j2\pi/N})e^{j(2\pi/N)n}+\frac{1}{2}H(e^{-j2\pi/N})e^{-j(2\pi/N)n}$$

$$=\frac{1}{2}\left(\frac{1}{1-\alpha e^{-j2\pi/N}}\right)e^{j(2\pi/N)n}+\frac{1}{2}\left(\frac{1}{1-\alpha e^{j2\pi/N}}\right)e^{-j(2\pi/N)n}$$

If we let $\quad \dfrac{1}{1-\alpha e^{-j2\pi/N}}=re^{j\theta}$

$$y[n]=r\cdot\cos(\frac{2\pi}{N}n+\theta)$$

The output of the sinusoidal input for LTI systems

$$x[n] = e^{j\frac{2\pi}{N}kn} = \cos\left(\frac{2\pi}{N}kn\right) + j\sin\left(\frac{2\pi}{N}kn\right)$$

$$\Rightarrow \quad y[n] = H(e^{j\frac{2\pi}{N}k}) \cdot e^{j\frac{2\pi}{N}kn}$$

$$H(e^{j\frac{2\pi}{N}k}) = \sum_{n=-\infty}^{\infty} h[n] \cdot \exp\left[-j\frac{2\pi}{N}kn\right]$$

$$\therefore \; y[n] = \left|H(e^{j\frac{2\pi}{N}k})\right| \exp\left[j\measuredangle H(e^{j\frac{2\pi}{N}k})\right] \cdot \exp\left[j\frac{2\pi}{N}kn\right]$$

$$= \left|H(e^{j\frac{2\pi}{N}k})\right| \exp\left[j\left(\frac{2\pi}{N}kn + \measuredangle H(e^{j\frac{2\pi}{N}k})\right)\right]$$

$$= \left|H(e^{j\frac{2\pi}{N}k})\right| \cos\left[\frac{2\pi}{N}kn + \measuredangle H(e^{j\frac{2\pi}{N}k})\right]$$

$$+ j\left|H(e^{j\frac{2\pi}{N}k})\right| \sin\left[\frac{2\pi}{N}kn + \measuredangle H(e^{j\frac{2\pi}{N}k})\right]$$

The output of the LTI systems using Fourier series

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t} \Rightarrow y(t)=\sum_{k=-\infty}^{\infty}a_k H(jk\omega_0)e^{jk\omega_0 t} \qquad \left(\omega_0=\frac{2\pi}{T}\right)$$

$$x[n]=\sum_{k=<N>}a_k e^{jk\omega_0 n} \Rightarrow y[n]=\sum_{k=<N}a_k H(e^{jk\omega_0})e^{jk\omega_0 n} \qquad \left(\omega_0=\frac{2\pi}{N}\right)$$

$$x(t)=A\cos(\omega_0 t+\theta_0) \Rightarrow y(t)=A|H(j\omega_0)|\cos[\omega_0 t+\theta_0+\angle H(j\omega_0)]$$

$$x(t)=A\sin(\omega_0 t+\theta_0) \Rightarrow y(t)=A|H(j\omega_0)|\sin[\omega_0 t+\theta_0+\angle H(j\omega_0)]$$

$$x[n]=A\cos(\omega_0 n+\theta_0) \Rightarrow y[n]=A|H(e^{j\omega_0})|\cos[\omega_0 n+\theta_0+\angle H(e^{j\omega_0})]$$

$$x[n]=A\sin(\omega_0 n+\theta_0) \Rightarrow y[n]=A|H(e^{j\omega_0})|\sin[\omega_0 n+\theta_0+\angle H(e^{j\omega_0})]$$

where $H(e^{j\omega})=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}, \qquad H(j\omega)=\int_{-\infty}^{\infty}h(t)e^{-j\omega t}dt$

3.9 Filtering

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

Filter

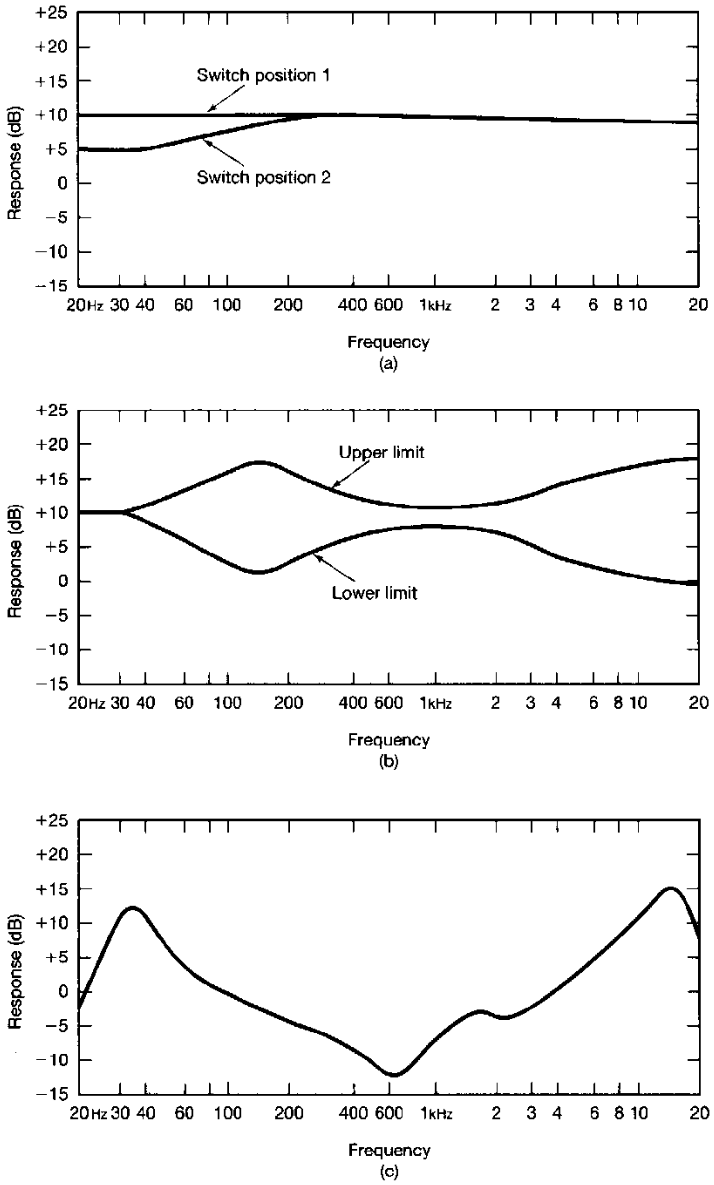
3.9.1 Frequency shaping filter

Change the shape of the spectrum

Ex: Equalizer

Bass control

Treble control



Differentiator

$$y(t) = \frac{dx(t)}{dt}$$

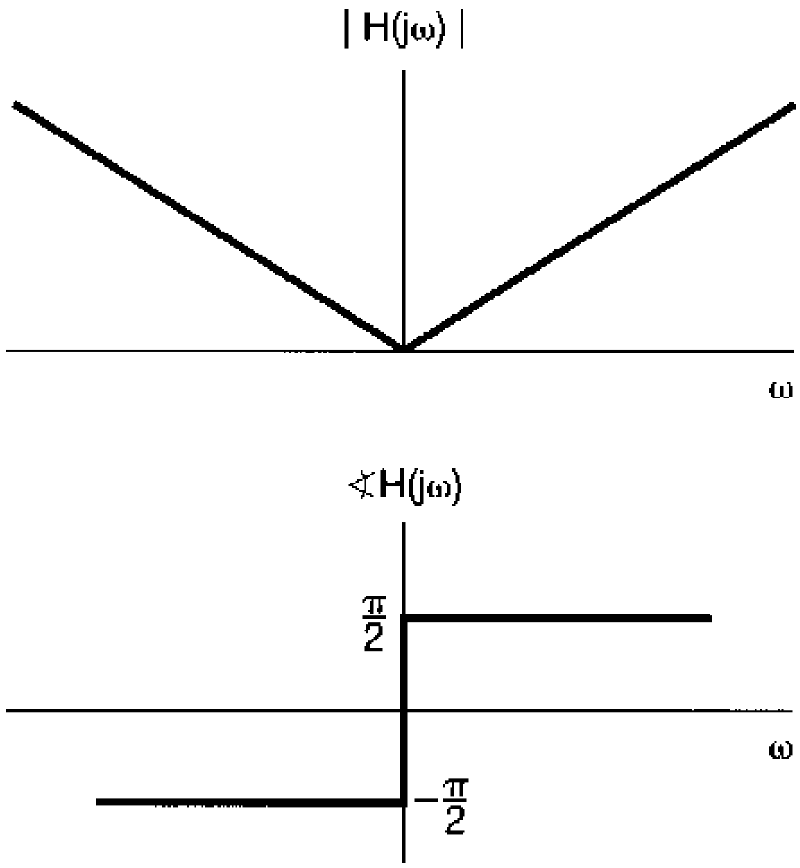
$$H(j\omega) = j\omega$$

- highpass filter

Ex.) Figure 3.24, page 235

$$\cos \omega t \rightarrow -\omega \sin \omega t = \omega \cos(\omega t + \frac{\pi}{2})$$

$$\sin \omega t \rightarrow \omega \cos \omega t = \omega \sin(\omega t + \frac{\pi}{2})$$



For Discrete-time systems

$$y[n] = x[n] - x[n-1]$$

$$x[n] = \sin \omega n$$

$$y[n] = \sin \omega n - \sin \omega (n-1)$$

$$= 2\cos(\omega n - \frac{\omega}{2})\sin(\frac{\omega}{2})$$

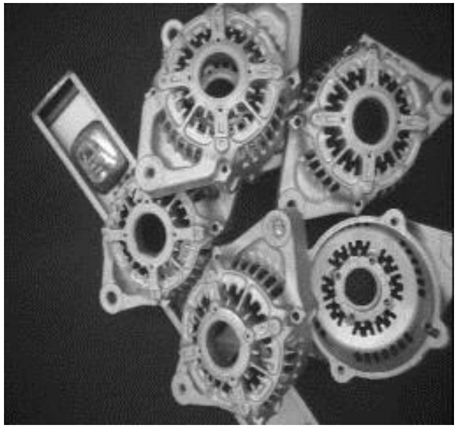
$$H(e^{j\omega}) = 1 - e^{-j\omega}$$

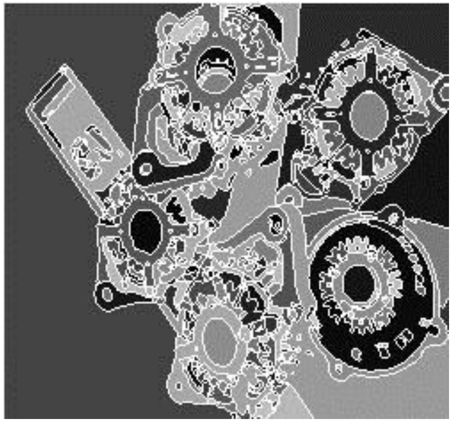
$$= e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})$$

$$= 2j\sin(\omega/2)e^{-j\omega/2}$$

Highpass filter

Edge detection





For Discrete-time systems

$$y[n] = \frac{1}{2}(x[n] + x[n-1])$$

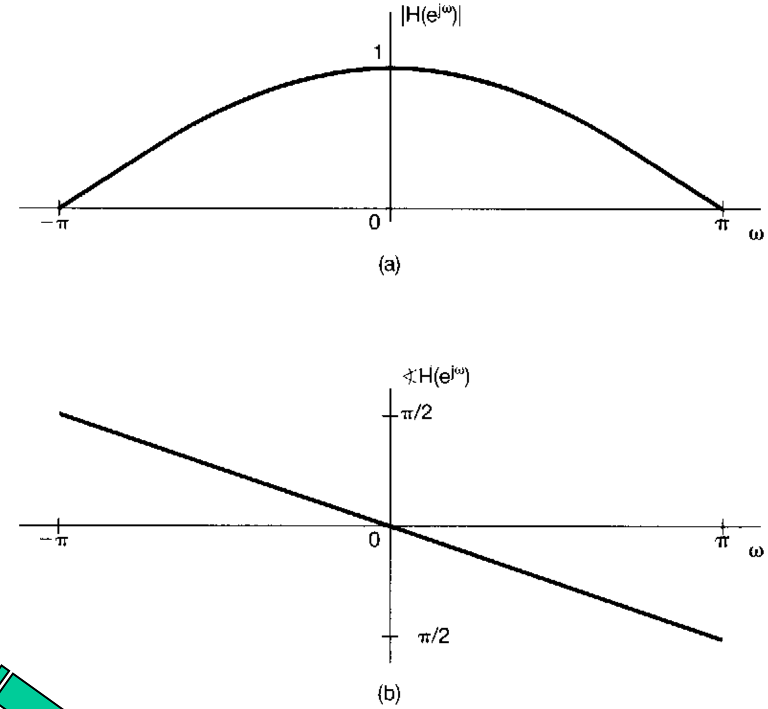
In frequency domain

$$H(e^{jw}) = \frac{1}{2}[1 + e^{-jw}] = e^{-jw/2}\cos(w/2)$$

In time domain, $x[n] = \sin \omega n$

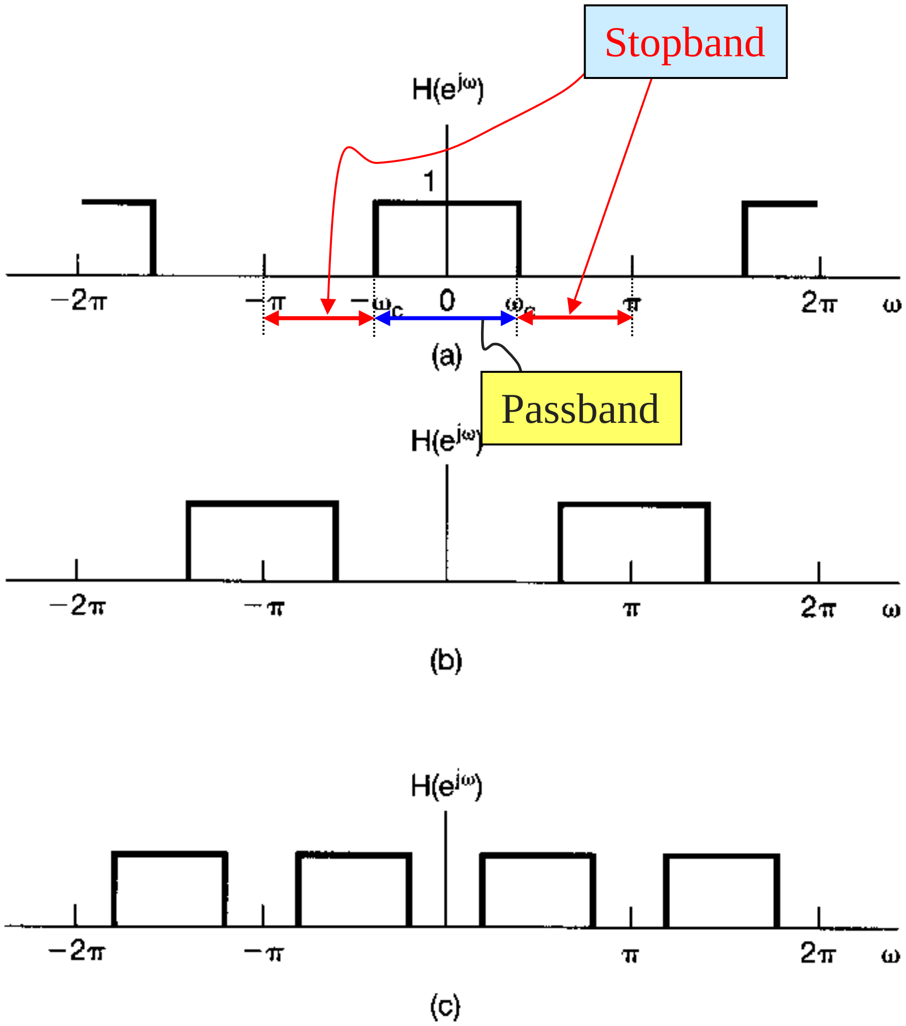
$$y[n] = \frac{1}{2}(\sin \omega n + \sin \omega(n-1))$$

$$= \sin(\omega n - \omega/2)\cos(\omega/2)$$



Lowpass filter

3.9.2 Frequency selective filters

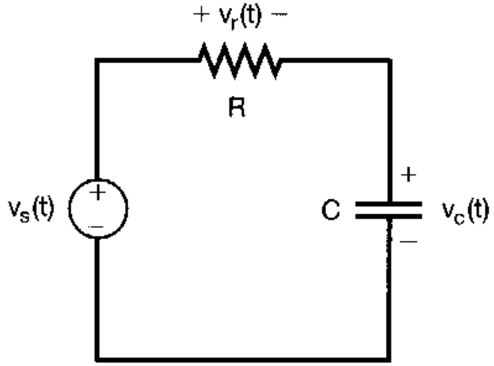


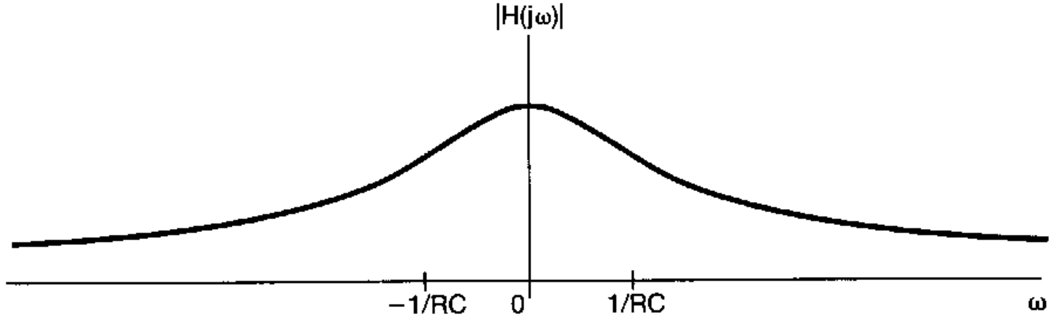
Lowpass filter

Highpass filter

Bandpass filter

3.10 Examples of continuous-time filters described by differential equations

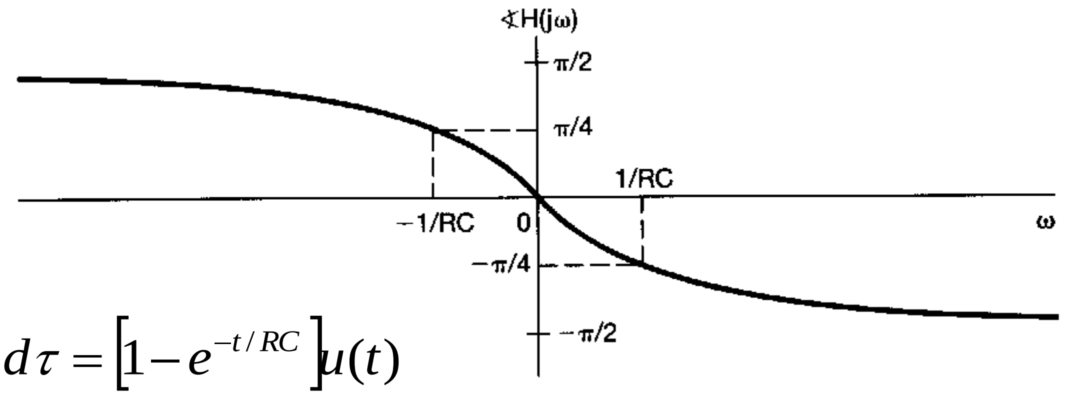




(a)

$$H(j\omega)=\frac{1}{1+RCj\omega}$$

$$h(t)=\frac{1}{RC}e^{-t/RC}u(t)$$

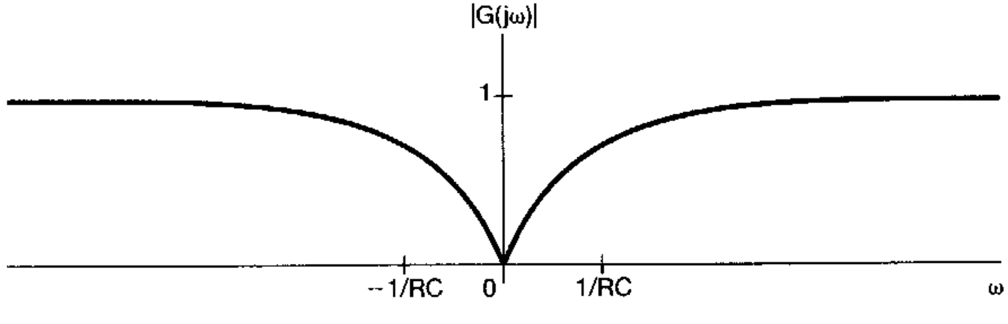


(b)

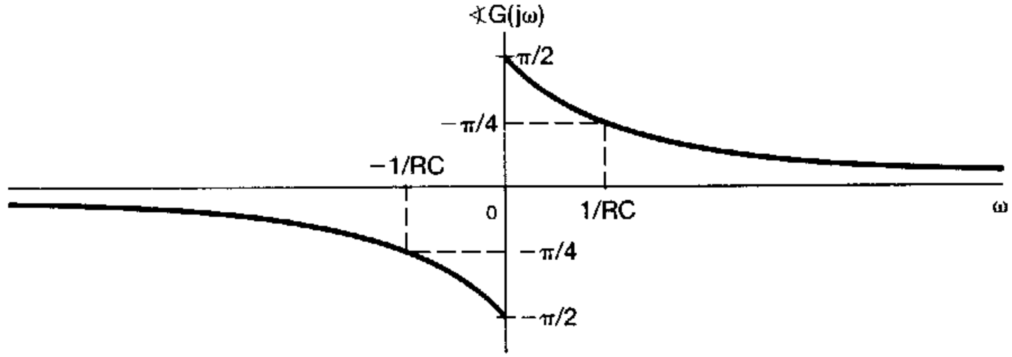
$$s(t)=u(t)*h(t)=\int_{-\infty}^{t}h(\tau)d\tau=\left[1-e^{-t/RC}\right]u(t)$$

For the voltage across the resistor R

$$G(jw) = \frac{jwRC}{1 + jwRC}$$



(a)



(b)

3.11 Examples of discrete-time filters described by difference equations

3.11.1 First-order recursive discrete-time filters

$$y[n] = ay[n-1] + x[n]$$

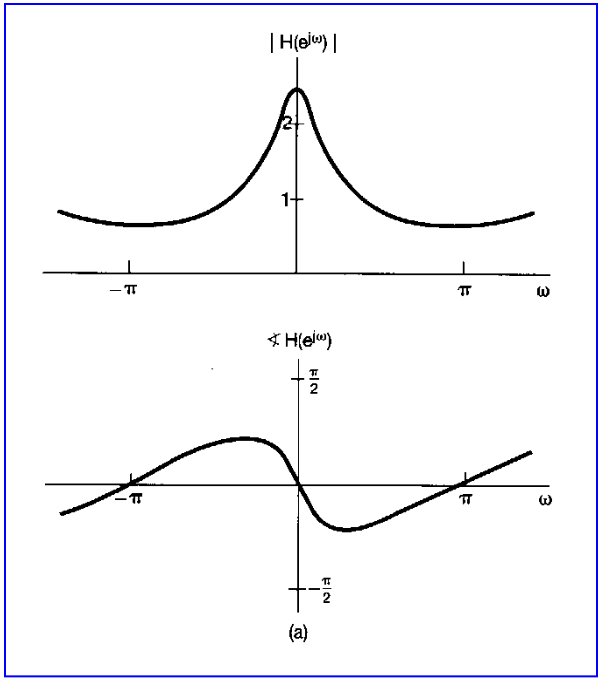
$$y[n] - ay[n-1] = x[n]$$

$$H(e^{jw})e^{jwn} - aH(e^{jw})e^{jw(n-1)} = e^{jwn}$$

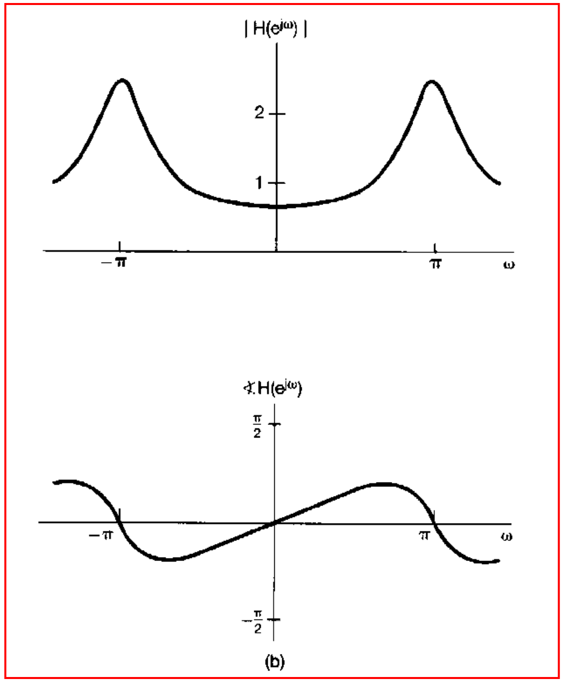
$$(1 - ae^{-jw})H(e^{jw})e^{jwn} = e^{jwn}$$

$$H(e^{jw}) = \frac{1}{(1 - ae^{-jw})}$$

$0 < a < 1$



$-1 < a < 0$



$$h[n] = a^n u[n]$$

$$H(e^{j\omega}) = 1 + ae^{-j\omega} + a^2 e^{-j2\omega} + \cdots$$

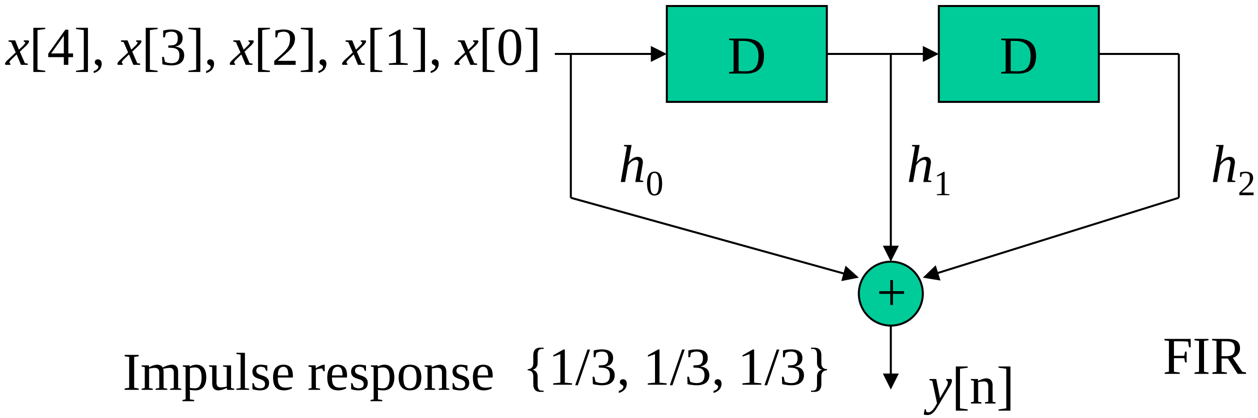
$$= \frac{1}{1 - ae^{-j\omega}}$$

$$s[n] = u[n] * h[n]$$: step response

$$= \frac{1 - a^{n+1}}{1 - a} u[n]$$

3.11.2 Nonrecursive discrete-time filters

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]$$ Moving Average



Impulse response {1/3, 1/3, 1/3}

FIR system (filter)

$$y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$$

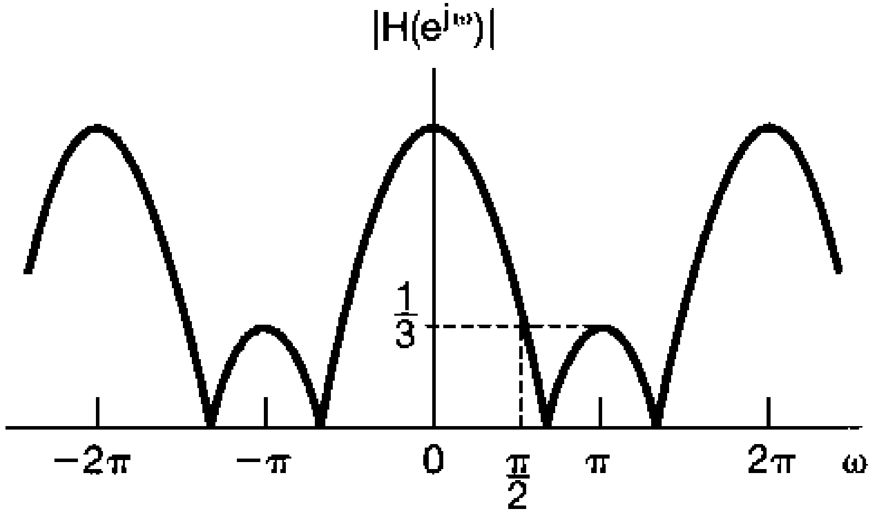
$$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$$

$$H(e^{j\omega}) = \frac{1}{3}[e^{j\omega} + 1 + e^{-j\omega}] = \frac{1}{3}(1 + 2\cos\omega)$$

$$H(e^{j\omega}) = \frac{1}{3}[1 + e^{-j\omega} + e^{-j2\omega}]$$

$$= \frac{1}{3}e^{-j\omega}(e^{j\omega} + 1 + e^{-j\omega})$$

$$= \frac{1}{3}e^{-j\omega}(1 + 2\cos\omega)$$



$\{h_0, h_1, h_2\} = \{1/4, 1/2, 1/4\}$

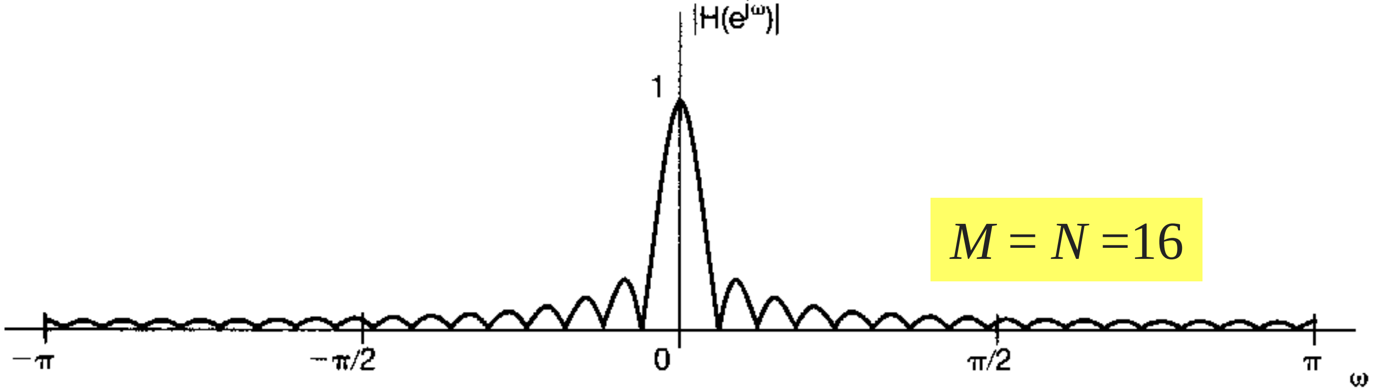
$$H(e^{j\omega}) = \frac{1}{4} + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} = e^{-j\omega}(\frac{1}{4}e^{j\omega} + \frac{1}{2} + \frac{1}{4}e^{-j\omega})$$

$$= \frac{1}{2}e^{-j\omega}(1 + \cos\omega)$$

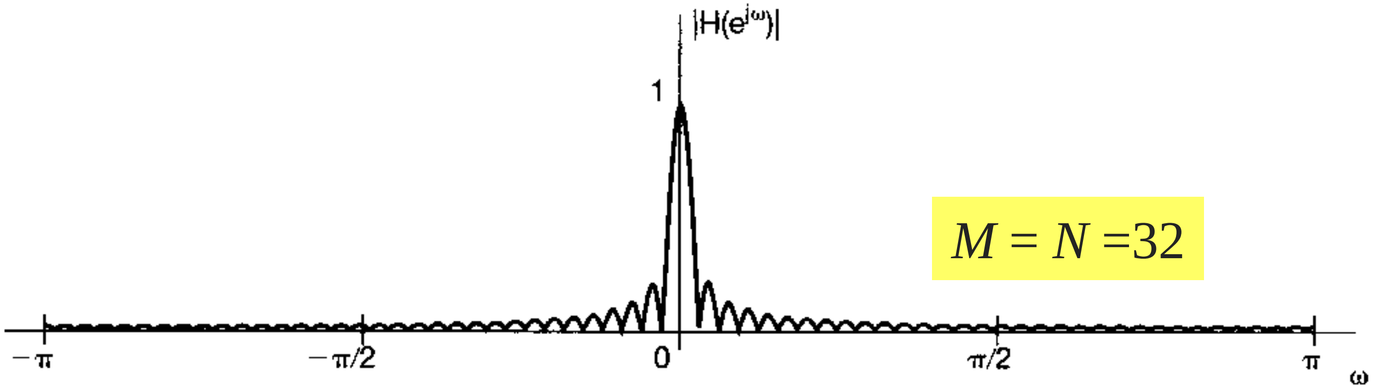
$$y[n] = \frac{1}{N+M+1} \sum_{k=-N}^{M} x[n-k]$$

$$H(e^{j\omega}) = \frac{1}{N+M+1} \sum_{k=-N}^{M} e^{-j\omega k}$$

$$H(e^{j\omega}) = \frac{1}{N+M+1} e^{-j\omega[(N-M)/2]} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}$$



(a)



(b)

$$y[n] = \frac{x[n] - x[n-1]}{2} \qquad H(e^{j\omega}) = \frac{1}{2}[1 - e^{-j\omega}] = je^{j\omega/2}\sin(\omega/2)$$