3

Fourier Series Representation of Periodic Signals

3.1 History

Reading

* Fourier series : Representing a periodic signal by a linear combination of harmonically related basic signals

* FFT : Cooley-Tukey (1965)
  Gauss !!

3.2 The response of LTI systems to complex exponentials

- For LTI systems & Continuous-time signals

e^{st} : Complex exponential input

$$e^{st} \rightarrow H(s)e^{st}$$

$e^{j\omega t}$: input $\qquad H(j\omega)e^{j\omega t}$: output

- For Discrete-time LTI systems

$$z^{n} \rightarrow H(z)z^{n}$$

Impulse response: $h(t)$, input : e^{st}, $e^{j\omega t}$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) x(t-\tau) d\tau = \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) x(t-\tau) d\tau = \int_{-\infty}^{+\infty} h(\tau) e^{j\omega(t-\tau)} d\tau$$

$$y(t) = e^{st} \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

$$y(t) = e^{j\omega t} \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$$y(t) = H(s) e^{st}$$

$$y(t) = H(j\omega) e^{j\omega t}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau$$

Laplace transform

Fourier transform

- Discrete-time systems

$$x[n] = z^n$$

z-transform

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

$$= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

$$y[n] = H(z)z^n$$

For a composite input

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

$$x(t) = \sum_k a_k e^{s_k t}$$

$$x[n] = \sum_k a_k z_k^n$$

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

$$y[n] = \sum_k a_k H(z_k) z_k^n$$

Ex) $y(t) = x(t-3)$

$\text{input}: x(t) = e^{j2t}$

$y(t) = e^{j2(t-3)} = e^{-j6} e^{j2t} \Rightarrow H(j2) = e^{-j6}$

$h(t) = ?\ \ h(t) = \delta(t-3)$

$$H(s) = \int_{-\infty}^{\infty} \delta(\tau - 3) e^{-s\tau} d\tau = e^{-3s}$$

$x(t) = \cos(4t) + \cos(7t)$

$y(t) = \cos(4(t-3)) + \cos(7(t-3))$

$$x(t) = \frac{1}{2} e^{j4t} + \frac{1}{2} e^{-j4t} + \frac{1}{2} e^{j7t} + \frac{1}{2} e^{-j7t}$$

$$y(t) = \frac{1}{2} H(j4) e^{j4t} + \frac{1}{2} H(-j4) e^{-j4t} + \frac{1}{2} H(j7) e^{j7t} + \frac{1}{2} H(-j7) e^{-j7t}$$

$$= \frac{1}{2} e^{-j12} e^{j4t} + \frac{1}{2} e^{j12} e^{-j4t} + \frac{1}{2} e^{-j21} e^{j7t} + \frac{1}{2} e^{j21} e^{-j7t}$$

$$= \frac{1}{2} e^{j4(t-3)} + \frac{1}{2} e^{-j4(t-3)} + \frac{1}{2} e^{j7(t-3)} + \frac{1}{2} e^{-j7(t-3)}$$

$$= \cos(4(t-3)) + \cos(7(t-3))$$

3.3 Fourier series representation of continuous-time periodic signals

3.3.1 Linear combinations of harmonically related complex exponentials

$x(t) = x(t+T)$ T : period (some positive value)

Fundamental period?

$\omega_0 = \dfrac{2\pi}{T}$: Fundamental frequency (when T is the fundamental period)

$$x(t) = \cos \omega_0 t$$

$$x(t) = e^{j\omega_0 t}$$

- Harmonically related complex exponentials

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$ *Fourier series* representation

$k = \pm 1$: first harmonic components

$k = \pm N$: N-th harmonic components

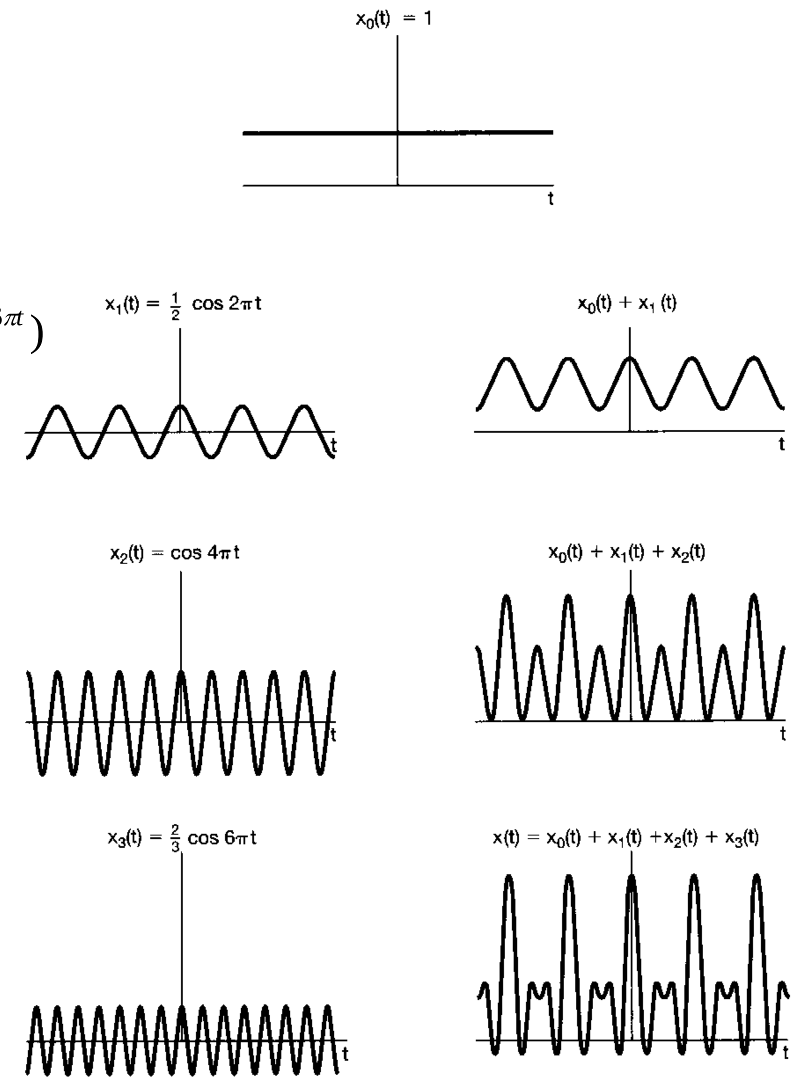
Ex. 3.2)

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}$$

$$x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t})$$

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

alternative form for the Fourier series
of real periodic signals



$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}=\cdots+a_{-2}e^{-j2\omega_0 t}+a_{-1}e^{-j\omega_0 t}+a_0+a_1 e^{j\omega_0 t}+a_2 e^{j2\omega_0 t}+\cdots$$

For $x(t)$ real

$$x(t)=x^*(t)$$

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad x^*(t)=\sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}$$

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}=\sum_{l=+\infty}^{-\infty} a_{-l} e^{-jl\omega_0 t}=\sum_{l=-\infty}^{+\infty} a_{-l} e^{-jl\omega_0 t}=\sum_{k=-\infty}^{+\infty} a_{-k} e^{-jk\omega_0 t}$$

$$\Longrightarrow \quad a_k^* = a_{-k}$$

- Alternative Representations

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}]$$

$$= a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}]$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\{a_k e^{jk\omega_0 t}\}$$

i) $a_k = A_k e^{j\theta_k}$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

ii) $a_k = B_k + jC_k$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t]$$

3.3.2 Determination of the Fourier series representation of a continuous-time signal

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Multiplying both sides by $e^{-jn\omega_0 t}$ and integrating over a period

$$\int_0^T x(t)e^{-jn\omega_0 t}dt=\int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}e^{-jn\omega_0 t}dt$$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt=\sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t}dt\right]$$

where $\int_0^T e^{j(k-n)\omega_0 t}dt=\int_0^T \cos(k-n)\omega_0 t\,dt + j\int_0^T \sin(k-n)\omega_0 t\,dt$

$$=\begin{cases} T & k=n \\ 0 & k\neq n \end{cases}$$

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

This is valid for any interval of length T

$$a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$ *synthesis* equation

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$ *analysis* equation

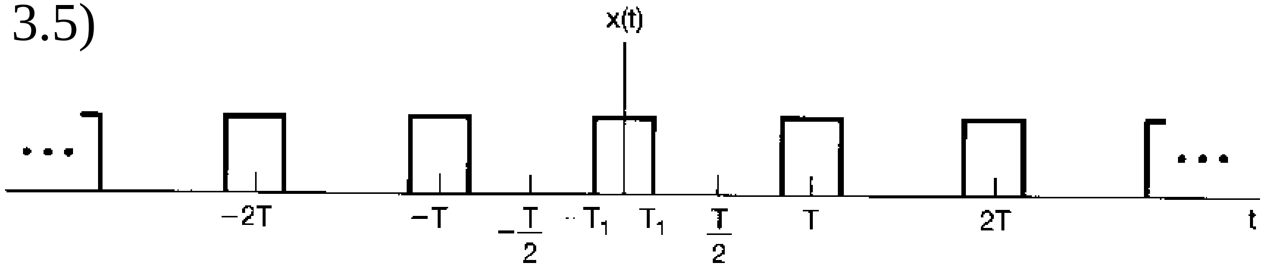
$\{a_k\}$ Fourier series coefficients of $x(t)$

or

Spectral coefficients of $x(t)$

$k=0,\quad a_0 = \frac{1}{T}\int_T x(t)dt$ DC or constant component of $x(t)$

Ex. 3.5)



$$x(t)=\begin{cases}1, & |t|<T_1\\ 0, & T_1<|t|<T/2\end{cases}$$

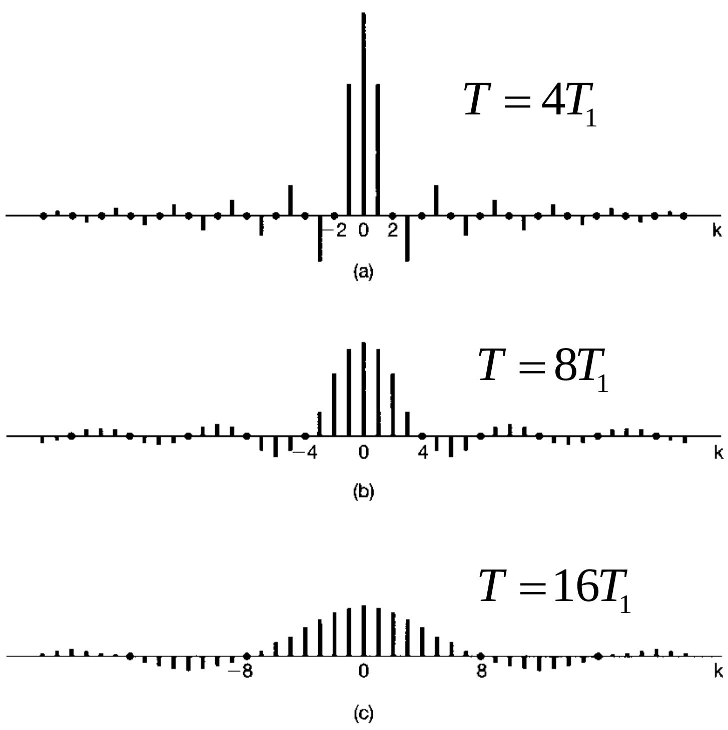
$$a_0=\frac{1}{T}\int_{-T_1}^{T_1}dt=\frac{2T_1}{T}$$

$$a_k=\frac{1}{T}\int_{-T_1}^{T_1}e^{-jk\omega_0 t}dt=-\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

$$=\frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1}-e^{-jk\omega_0 T_1}}{2j}\right]$$

$$\therefore\quad a_k=\frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}=\frac{\sin(k\omega_0 T_1)}{k\pi},k\neq 0$$

Plots of Ta_k



3.4 Convergence of the Fourier series

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$ Truncated approximation up to N

$$E_N = \int_T |e_N(t)|^2 dt$$

Determination of coefficients : Minimizing the energy of approximation error

solution

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$ (Prob. 3.66)

identical to the expression used to determine the Fourier series coefficients

Condition for the guarantee of Fourier series representation

For square integrable signal

$$\int_T |x(t)|^2 dt < \infty,$$

the energy of the approximation error converges to zero as the order increases.

$$e(t) = x(t) - \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Longrightarrow \int_T |e(t)|^2 dt = 0$$

This does not imply that

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{for all } t.$$

Convergence in Mean Square Error (MSE) Sense!!

Alternative condition for the guarantee of Fourier series representation

- Dirichlet condition

1. Over any period, x(t) must be *absolutely integrable*.

$$\int_T |x(t)| \ dt < \infty$$

Example of the function that is not absolutely integrable

$$x(t) = \frac{1}{t}, \qquad 0 < t \leq 1$$

2. Finite number of maxima and minima
   - $x(t)$ is of bounded variation.

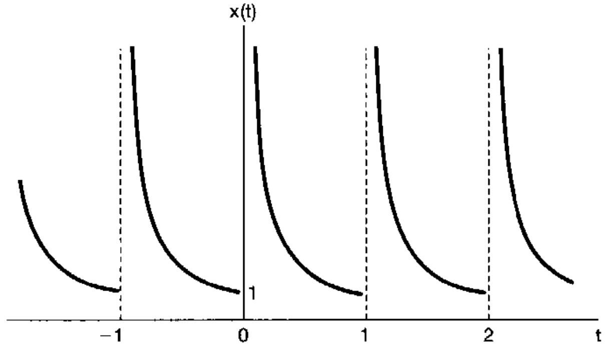
$$x(t) = \sin(2\pi / t), \qquad 0 < t \leq 1$$

3. Finite number of discontinuities

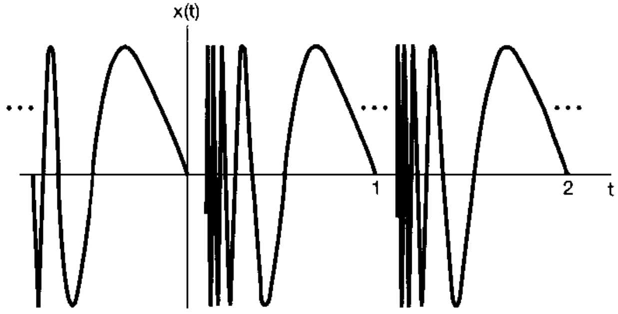
Examples that do not meet Dirichlet conditions

$$x(t) = \frac{1}{t}, \qquad 0 < t \le 1$$

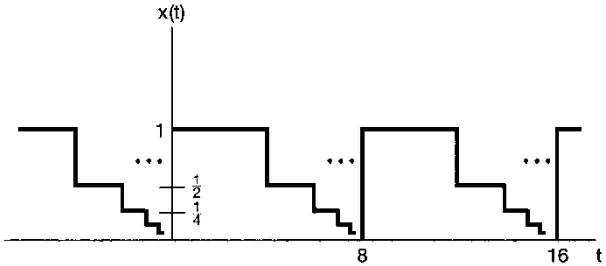
$$x(t) = \sin(2\pi / t), \qquad 0 < t \le 1$$



(a)



(b)



$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jkw_0 t}$$

Mismatches in discontinuous points :
Gibbs Phenomena