2

LINEAR TIME-INVARIANT SYSTEMS

2.0 INTRODUCTION

- In continuous time, linear time-invariant systems are rare.
- In a short term viewpoint, these assumptions are permissible.
- In discrete time, theses systems can be constructed (e.g. implemented in a digital computer).

Linearity : Superposition Property

If an input can be represented by a linear combination of basic signals, the output can also be represented by the same linear combination of outputs corresponding to the individual basic input signals.

Time-invariance : Shift-invariance

2.1.1 The representation of discrete-time signals in terms of impulses

Given an input sequence $x[n]$

$$x[-1]\delta[n+1]=\begin{cases} x[-1] & n=-1 \\ 0 & n\neq -1 \end{cases}$$

$$x[0]\delta[n]=\begin{cases} x[0] & n=0 \\ 0 & n\neq 0 \end{cases}$$

$$x[1]\delta[n-1]=\begin{cases} x[1] & n=1 \\ 0 & n\neq 1 \end{cases}$$

$$\begin{aligned} x[n] &= \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] \\ &\quad + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots \end{aligned}$$

$x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$; sifting property of the discrete-time unit impulse

(Fig.2.1, Page 76)

2.1.2 The Discrete-Time Unit Impulse Response and the Convolution Sum Representation of LTI systems

Let $h_k[n]$ be the output of the linear system corresponding to the input $\delta[n-k]$

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k] \Longrightarrow y[n]=\sum_{k=-\infty}^{\infty}x[k]h_k[n]$$

For a time-invariant system, $h_k[n]=h_0[n-k]=h[n-k]$

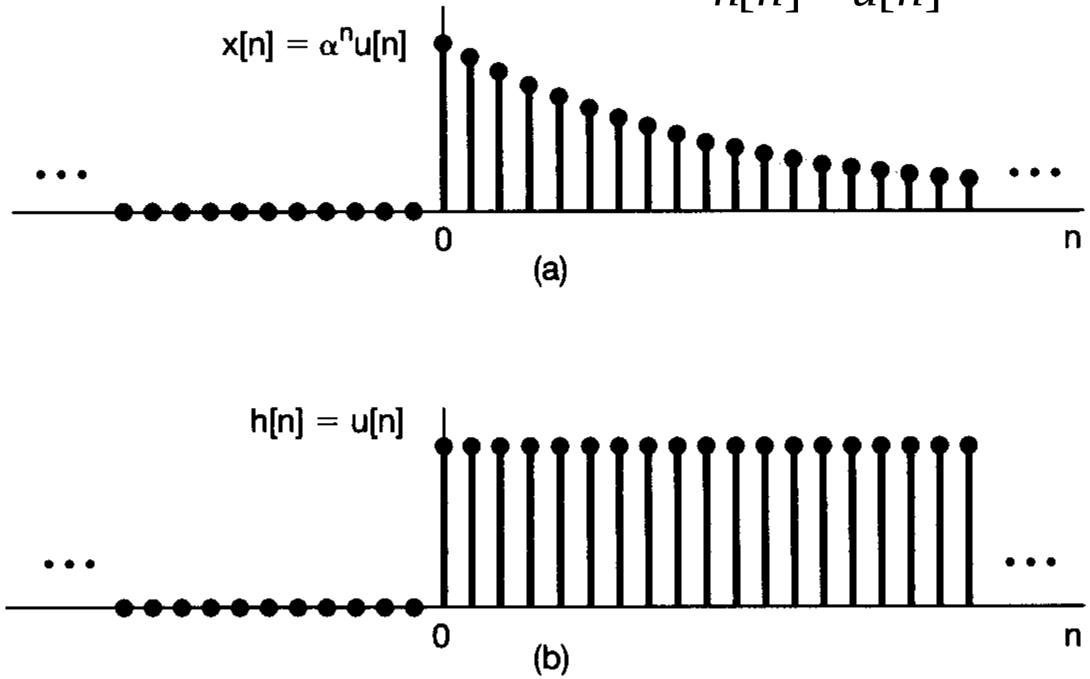
$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k] \quad : \quad \text{Convolution Sum}$$

$$y[n]=x[n]*h[n]$$

Example 2.3

$$x[n]=\alpha^n u[n] \quad , \quad |\alpha|<1$$

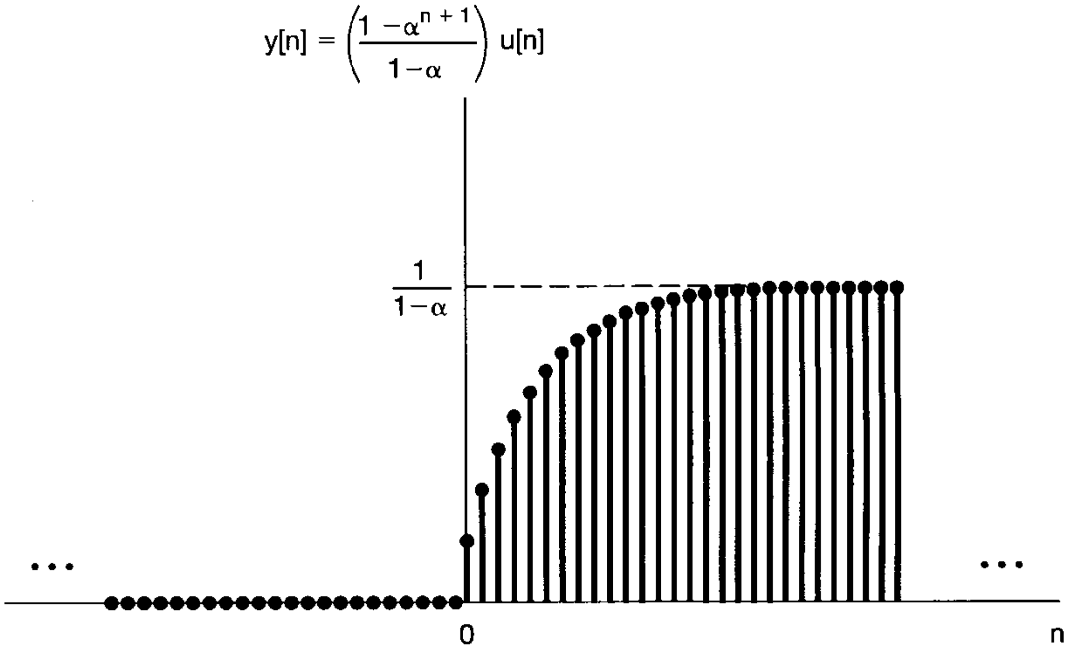
$$h[n]=u[n]$$



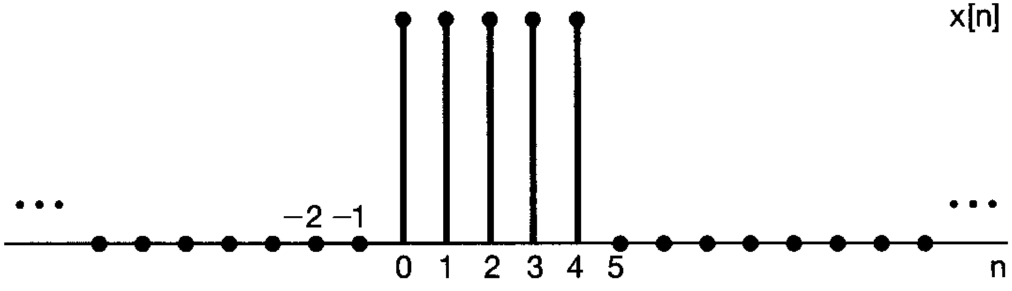
$$y[n]=\sum_{k=-\infty}^{\infty}\left(\alpha^k u[k]\right)\cdot u[n-k]=\sum_{k=0}^{n}\alpha^k=\frac{1-\alpha^{n+1}}{1-\alpha} \qquad for \quad n\geq 0$$

:Graphical interpretation (Fig.2.6, Page 84)

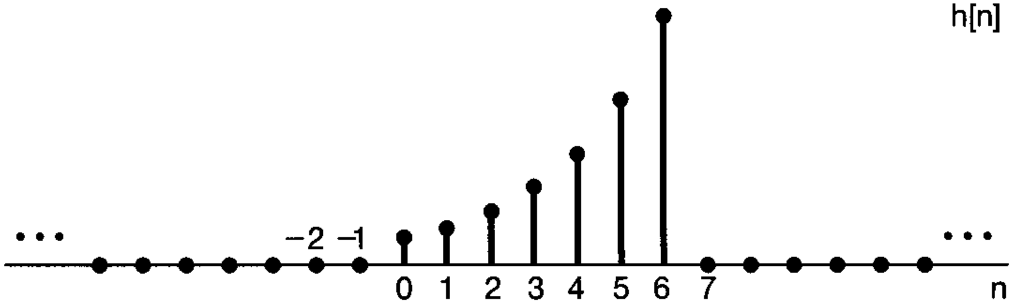
$$y[n]=\left(\frac{1-\alpha^{n+1}}{1-\alpha}\right)u[n]$$



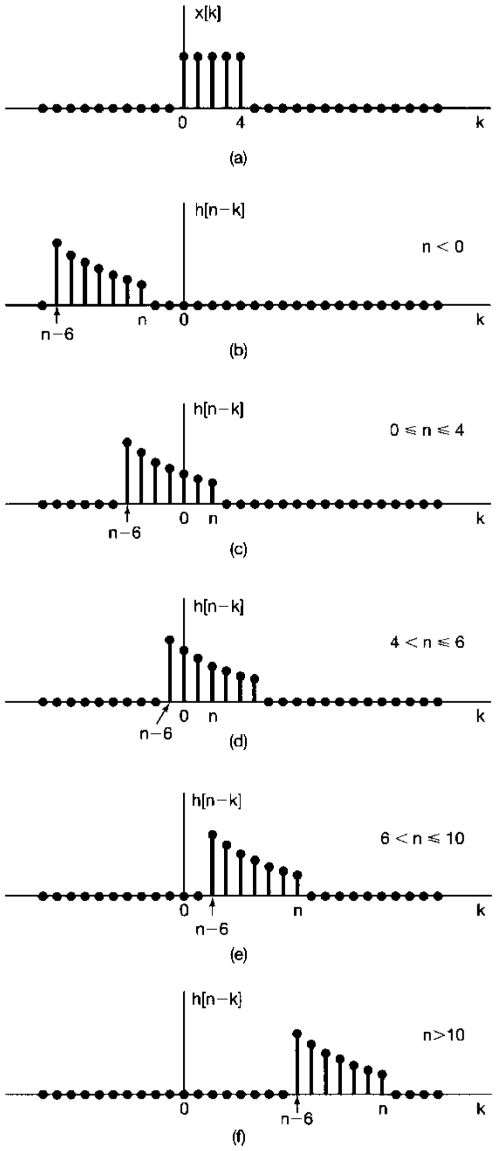
Example 2.4 $x[n]=\begin{cases}1, & 0\le n\le 4\\ 0, & otherwise\end{cases}$ $h[n]=\begin{cases}\alpha^n, & 0\le n\le 6\\ 0, & otherwise\end{cases}$

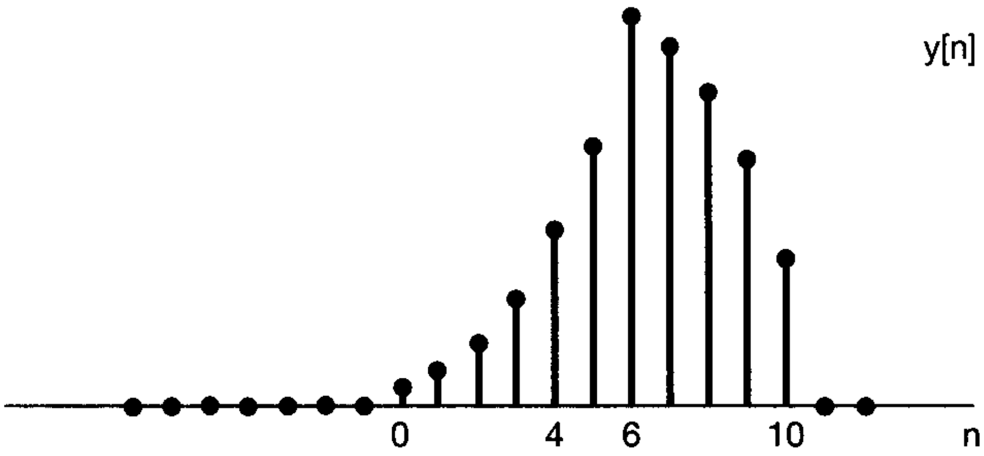


(a)

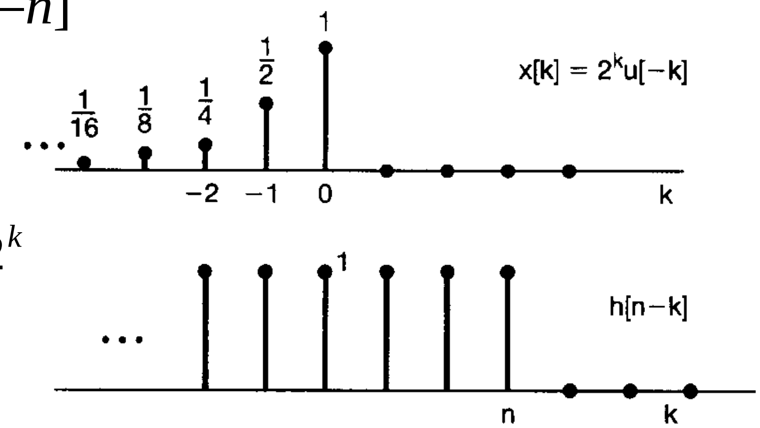


(b)


x[k]
0
4
k
(a)
h[n−k]
n < 0
n−6
n
0
k
(b)
h[n−k]
0 ≤ n ≤ 4
n−6
0
n
k
(c)
h[n−k]
4 < n ≤ 6
n−6
0
n
k
(d)
h[n−k]
6 < n ≤ 10
0
n−6
n
k
(e)
h[n−k]
n>10
0
n−6
n
k
(f)


y[n]
0
4
6
10
n

Example 2.5 $\quad h[n]=u[n] \qquad x[n]=2^n u[-n]$



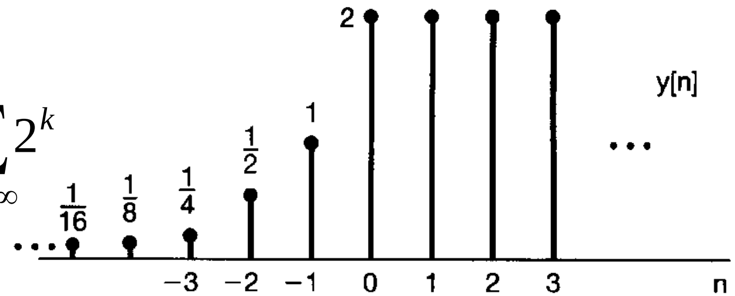
(a)

For $n>=0$

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{\infty}2^k u[-k]u[n-k]=\sum_{k=-\infty}^{0}2^k$$

$$=\sum_{r=0}^{\infty}\left(\frac{1}{2}\right)^r=\frac{1}{1-(1/2)}=2$$

For $n<0$



(b)

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{\infty}2^k u[-k]u[n-k]=\sum_{k=-\infty}^{n}2^k$$

$$=\left(\frac{1}{2}\right)^{-n}\sum_{m=0}^{\infty}\left(\frac{1}{2}\right)^m=2^n\cdot 2=2^{n+1}$$

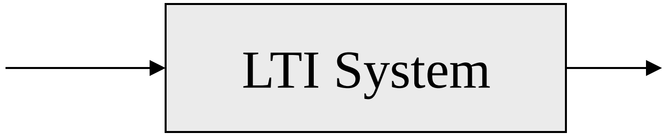
$m=-k$

2.2 Continuous-time LTI systems : the convolution integral

$\delta(t)$ → $h(t)$

$\delta(t-\tau)$ → $h(t-\tau)$

$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$ → $\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$



Sifting property of the continuous-time impulse

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

the value of the function evaluated at some point

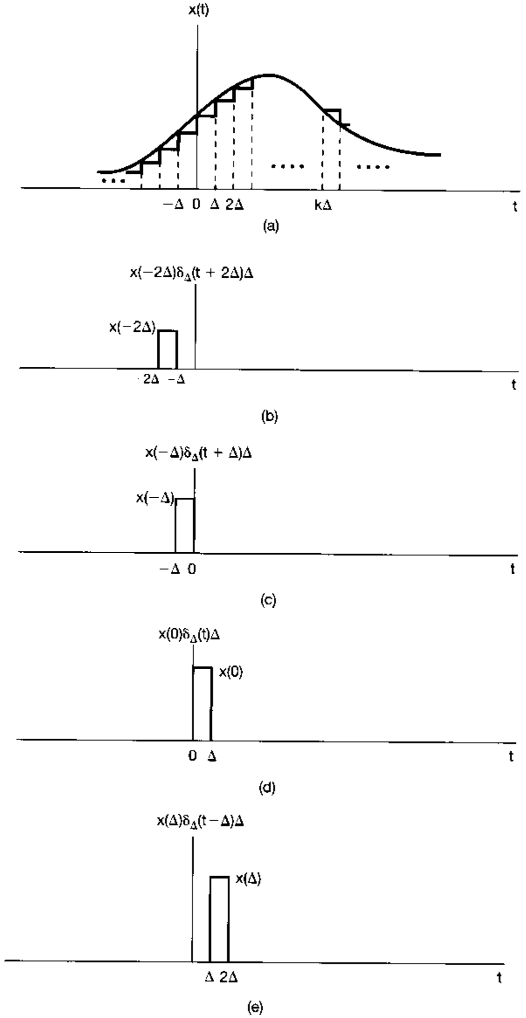
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(\tau-t)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

a delta function with a sterngth
equal to the value of $x(t)$ evaluated at $t = t_0$

cf) Sampling property: $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$

2.2.1 The representation of continuous-time signals in terms of impulses



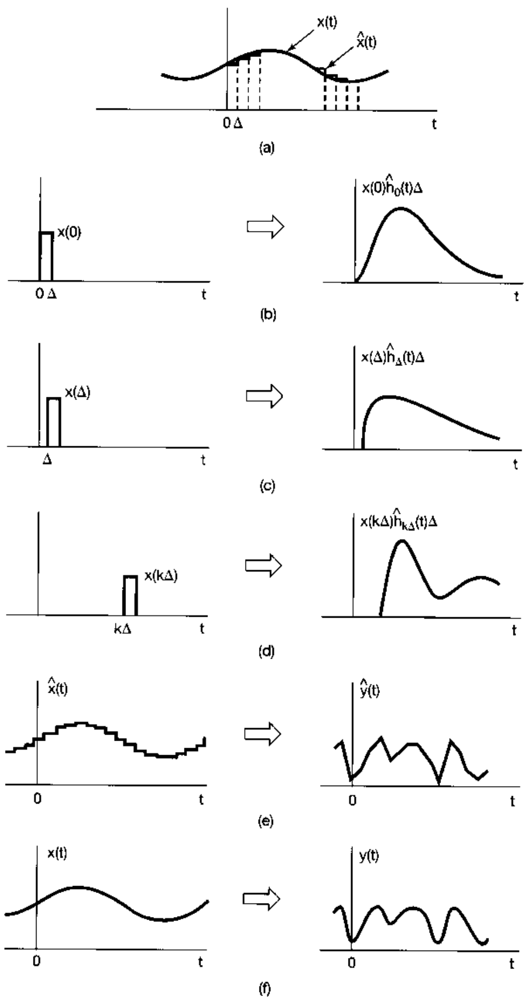
$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & otherwise \end{cases}$$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

2.2.2 The continuous-time unit impulse response and the convolution integral representation of LTI systems



$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

Linear

$$\hat{y}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)h_{k\Delta}(t)\Delta$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h_{\tau}(t)d\tau$$

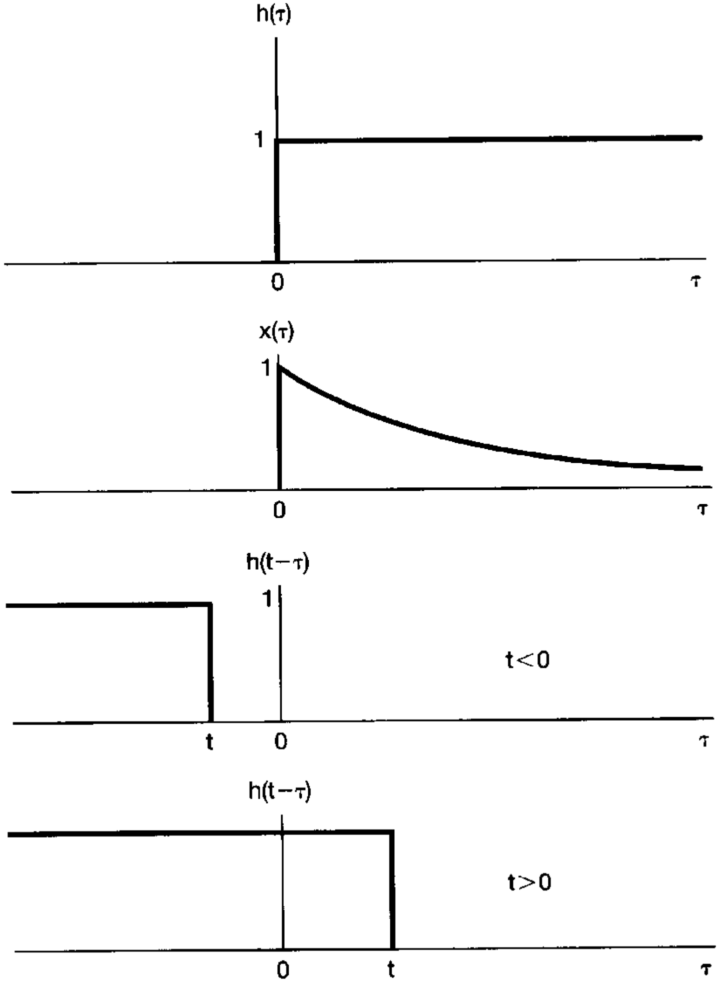
For an LTI system $h_{\tau}(t) = h(t - \tau)$

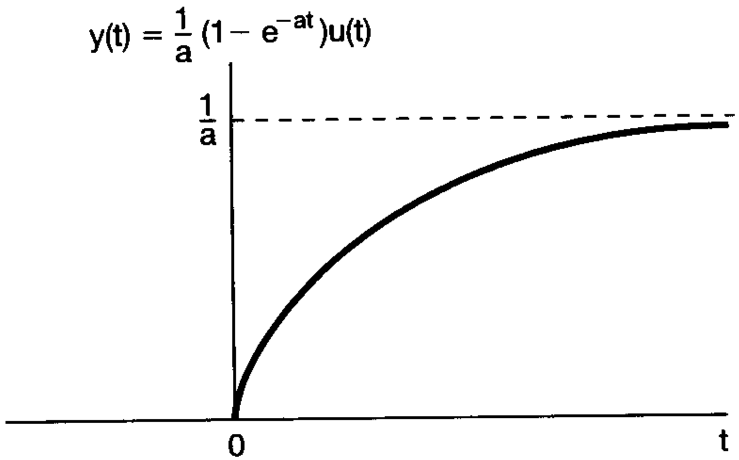
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t - \tau)d\tau$$

$y(t) = x(t) * h(t)$; convolution integral
or superposition integral

Example 2.6 $x(t) = e^{-at}u(t), \qquad a > 0$

$h(t) = u(t)$





Note) step response of an LPF,

when $h(t)$: input,

$x(t)$: impulse response of an LPF

Example 2.7

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & otherwise \end{cases} \qquad h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & otherwise \end{cases}$$