- Harmonically related complex exponentials

Sets of periodic exponentials with a common period T_0 or fundamental frequencies that are all multiples of a single positive frequency ω_0:

$$\phi_k(t) = e^{jk\omega_0 t}, k = 0, \pm 1, \pm 2, \ldots \qquad \text{where } \omega_0 = \frac{2\pi}{T_0}$$

✓ Fundamental period: $\dfrac{2\pi}{|k|\omega_0} = \dfrac{T_0}{|k|}$

✓ Fundamental frequency: $|k|\omega_0$

Example 1.5: sum of two complex exponentials

$\Rightarrow$ product of a single complex exponential and a single sinusoid

$$x(t) = e^{j2t} + e^{j3t}$$
$$= e^{j2.5t}(e^{-j0.5t} + e^{j0.5t})$$
$$= 2e^{j2.5t}\cos(0.5t)$$

$$|x(t)| = 2|\cos(0.5t)|$$

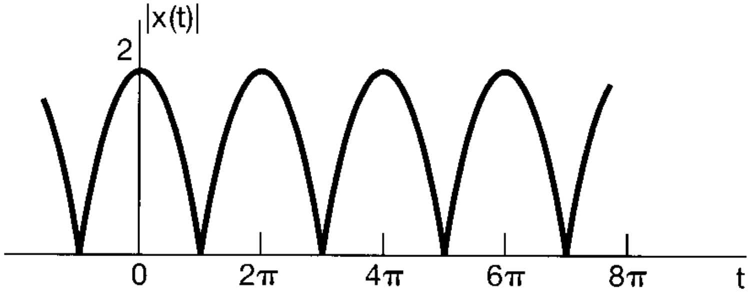


Figure 1.22 The full-wave rectified sinusoid of Example 1.5.

- *General complex exponential signals*

$$C, a : \text{complex number} \qquad C = |C|e^{j\theta} \qquad a = r + j\omega_0$$

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

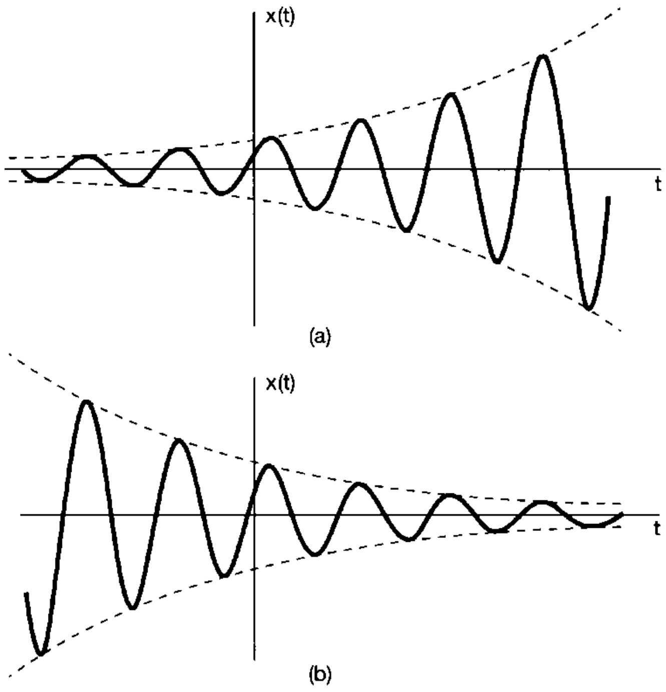


Figure 1.23 (a) Growing sinusoidal signal $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r > 0$; (b) decaying sinusoid $x(t) = Ce^{rt}\cos(\omega_0 t + \theta)$, $r < 0$.

1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals (sequence)

$$x[n] = C\alpha^n, \quad C \;\&\; \alpha \text{ :complex numbers}$$

$$= Ce^{\beta n}$$

cf.) $x(t) = Ce^{at}$

- *Real exponential signals*

  $C \;\&\; \alpha$: real numbers

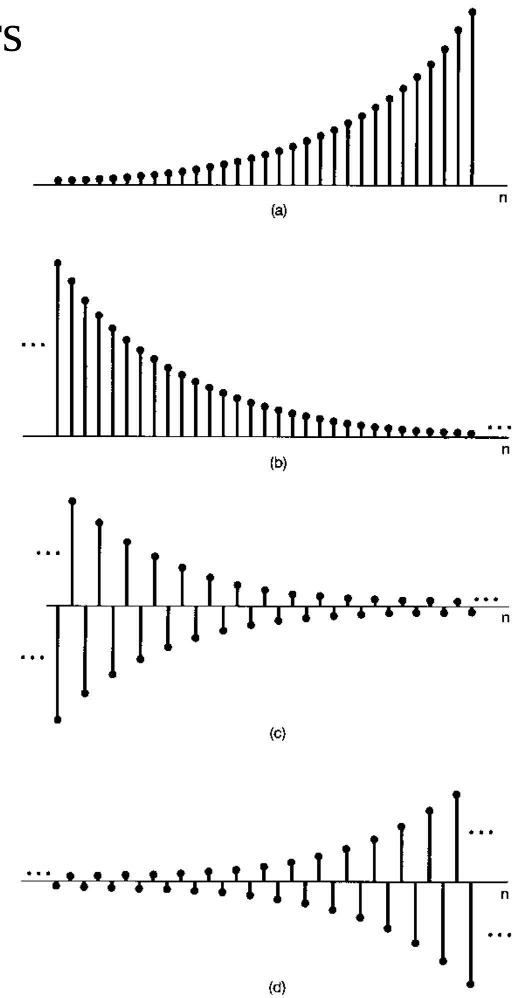


Figure 1.24 The real exponential signal $x[n] = C\alpha^n$: (a) $\alpha > 1$; (b) $0 < \alpha < 1$; (c) $-1 < \alpha < 0$; (d) $\alpha < -1$.

- *Sinusoidal signals*

$$x[n] = e^{j\omega_0 n} \qquad (1)$$

$$x[n] = A\cos(\omega_0 n + \phi) \qquad (2)$$

$$e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n)$$

$$A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

- (1), (2): Infinite total energy
but finite average power

How about Periodic ?

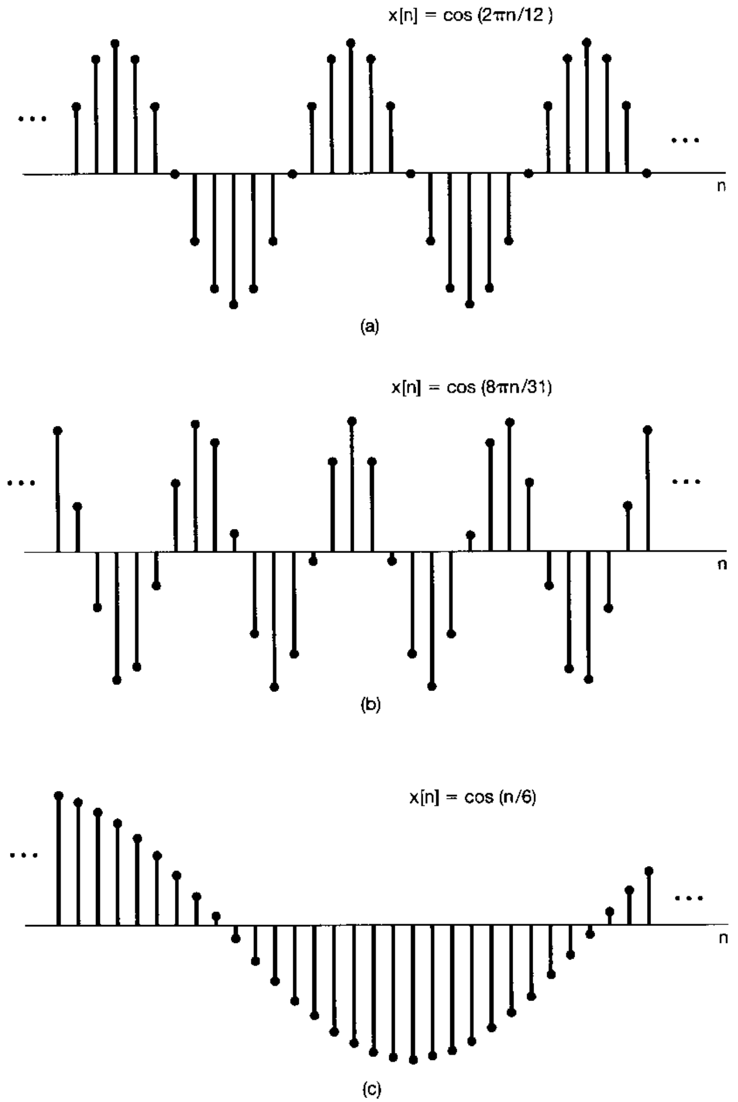


Figure 1.25 Discrete-time sinusoidal signals.

• *General complex exponential signals*

$$C = |C|e^{j\theta} \quad \alpha = |\alpha|e^{j\omega_0}$$

$$C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)$$

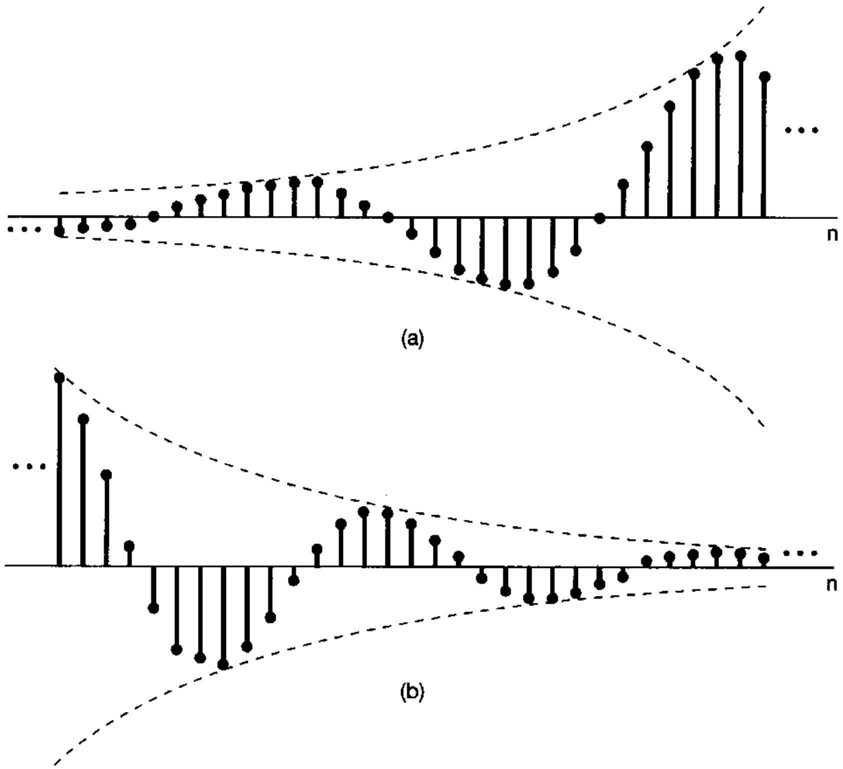


Figure 1.26 (a) Growing discrete-time sinusoidal signals; (b) decaying discrete-time sinusoid.

1.3.3 Periodicity properties of discrete-time complex exponentials

- Continuous-time signal $\cos(\omega_0 t)$

$$-\infty < \omega_0 < \infty$$

- Discrete-time signal $\cos(\omega_0 n)$

$$\cos\big((\omega_0 + 2\pi)n\big) = \cos\big(\omega_0 n\big)$$

$$0 \le \omega_0 < 2\pi \ \ (-\pi \le \omega_0 < \pi)$$

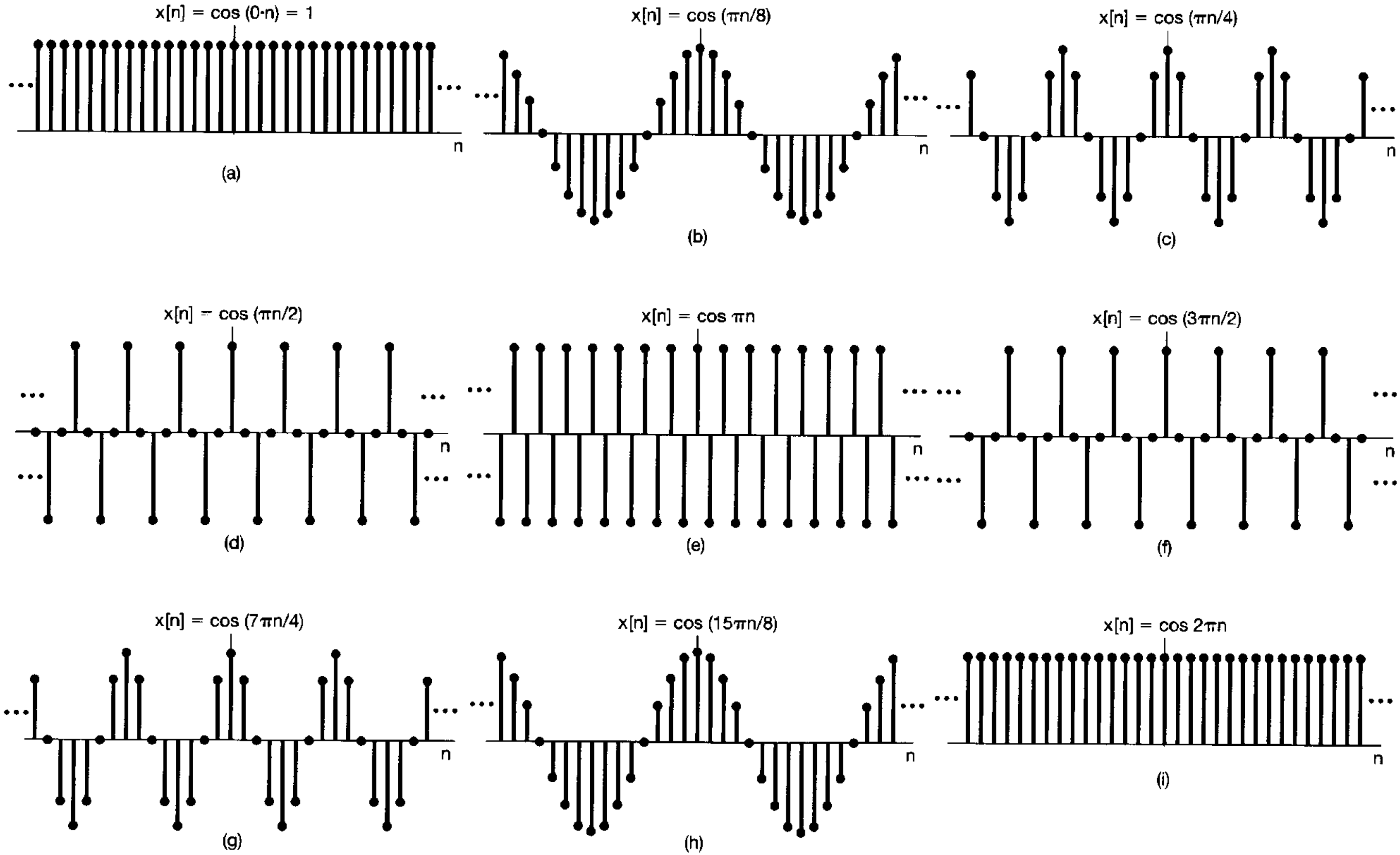


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n} e^{j\omega_0 n} = e^{j\omega_0 n}$$

$$e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

$$0 \le \omega_0 < 2\pi \;\; (-\pi \le \omega_0 < \pi)$$

* Period of a discrete-time signal

Find N such that $e^{j\omega_0 (n+N)} = e^{j\omega_0 n}$ or $e^{j\omega_0 N} = 1$

$$\omega_0 N = 2\pi m$$

$$\frac{\omega_0}{2\pi} = \frac{m}{N}$$

$$N = m\left(\frac{2\pi}{\omega_0}\right)$$

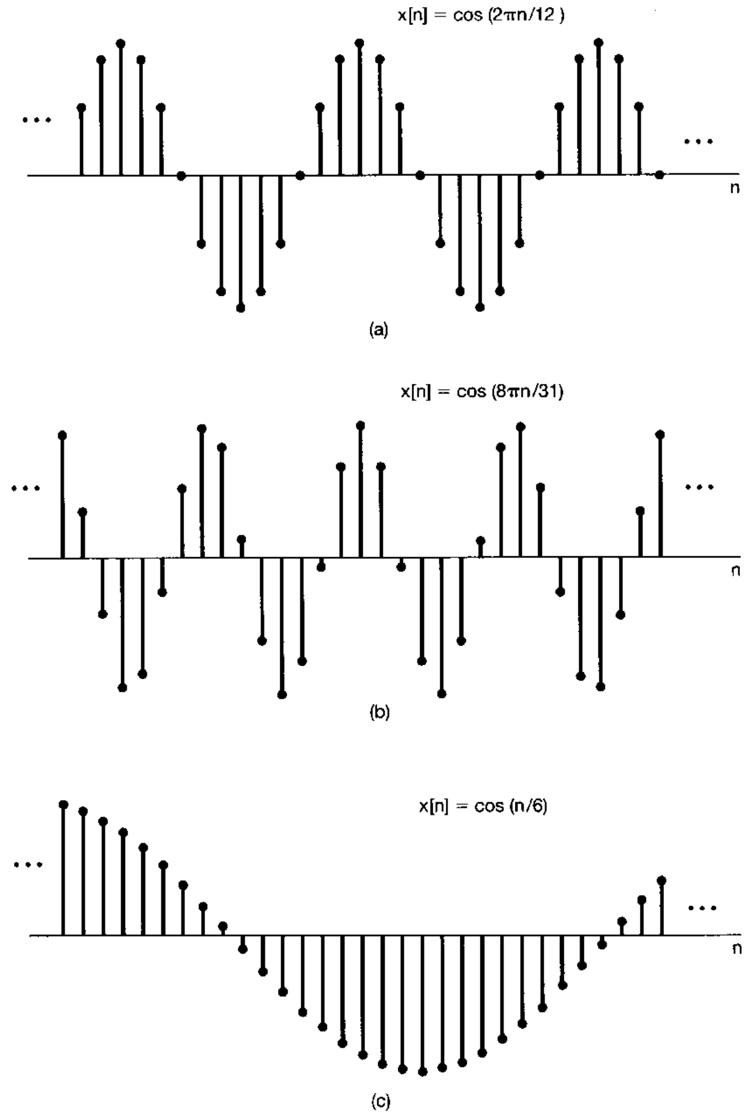


Figure 1.25 Discrete-time sinusoidal signals.

$$\omega_0 = \frac{2\pi}{12} \qquad \frac{\omega_0}{2\pi} = \frac{1}{12}$$

$$\omega_0 = \frac{8\pi}{31} \qquad \frac{\omega_0}{2\pi} = \frac{4}{31}$$

$$\omega_0 = \frac{1}{6} \qquad \frac{\omega_0}{2\pi} = \frac{1}{12\pi}$$

- Harmonically related periodic exponentials
  (periodic exponentials with a common period N)

$$\phi_k[n] = e^{jk(2\pi/N)n}, k = 0, \pm 1, \ldots \quad \text{cf.)} \ \phi_k(t) = e^{jk\omega_0 t}, k = 0, \pm 1, \pm 2, \ldots$$

$$\phi_{k+N}[n] = e^{j(k+N)(2\pi/N)n} = e^{jk(2\pi/N)n} e^{j2\pi n} = e^{jk(2\pi/N)n} = \phi_k[n]$$

$$\phi_0[n] = 1, \phi_1[n] = e^{j2\pi n/N}, \phi_2[n] = e^{j4\pi n/N}, \ldots, \phi_{N-1}[n] = e^{j2\pi(N-1)n/N}$$

Meaningful only for $k = 0, 1, 2, \cdots, N-1$

$$\phi_N[n] = \phi_0[n], \ \phi_{-1}[n] = \phi_{N-1}[n]$$

1.4 The Unit Impulse and Unit Step Function

1.4.1 The discrete-time unit impulse and unit step sequences

- Discrete-time unit impulse

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

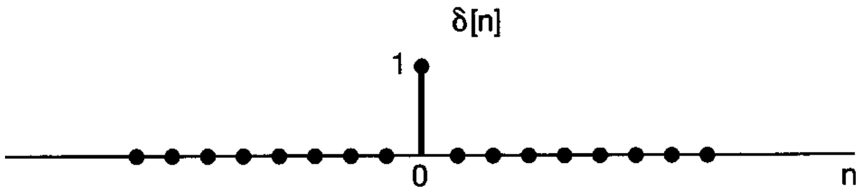


Figure 1.28 Discrete-time unit impulse (sample).

- Discrete-time unit step

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

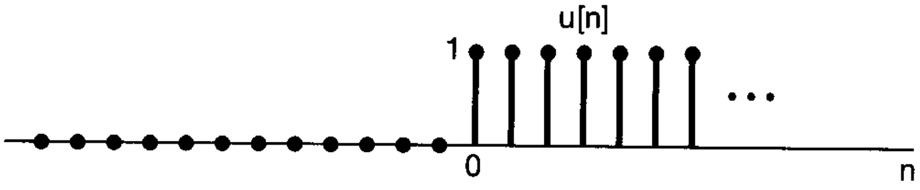


Figure 1.29 Discrete-time unit step sequence.

✓ $\delta[n] = u[n] - u[n-1]$; the *first difference* of the discrete-time step

✓ $u[n] = \sum_{m=-\infty}^{n} \delta[m]$; the *running sum* of the unit sample

$$\Leftrightarrow u[n] = \sum_{k=\infty}^{0} \delta[n-k]$$

$$\Leftrightarrow u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$; a superposition of delayed impulses (Fig. 1.31, p. 32)

✓ $x[n]\delta[n] = x[0]\delta[n]$

✓ $x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$; Sampling property of the unit impulse

1.4.2 The continuous-time unit step and unit impulse functions

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

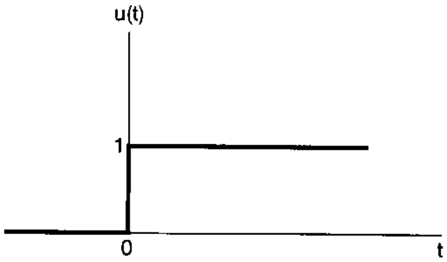


Figure 1.32 Continuous-time unit step function.

✓ $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$; the *running integral* of the unit impulse (Fig. 1.37, p.35)

✓ $\delta(t) = \dfrac{du(t)}{dt}$; the *first derivative* of the continuous-time unit step

✓ Interpretation of $\delta(t) = \dfrac{du(t)}{dt}$

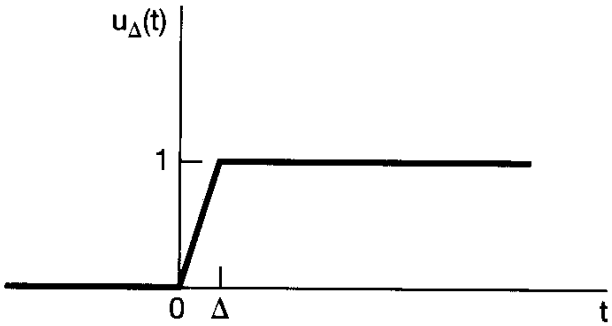


Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

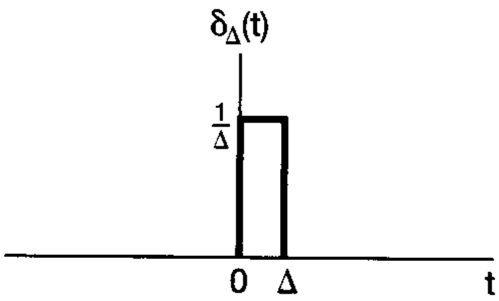


Figure 1.34 Derivative of $u_\Delta(t)$.

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt} \xrightarrow{\Delta \to 0} \delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

$$\int_{-\infty}^{t} k\delta(\tau)d\tau = ku(t)$$

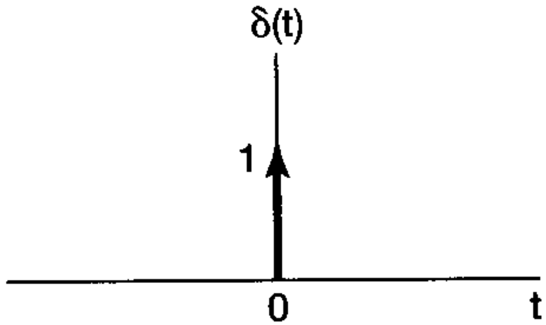


Figure 1.35 Continuous-time unit impulse.

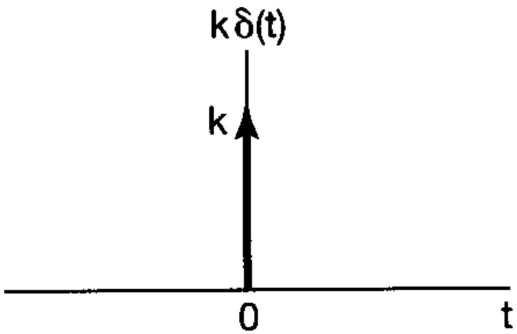


Figure 1.36 Scaled impulse.

✓ Graphical interpretation (Fig. 1.38, p. 35)

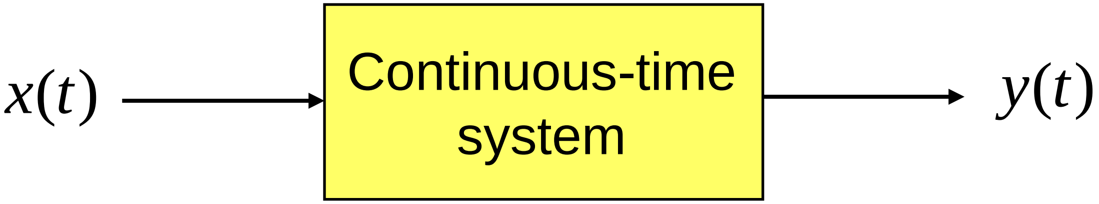
$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma)$$

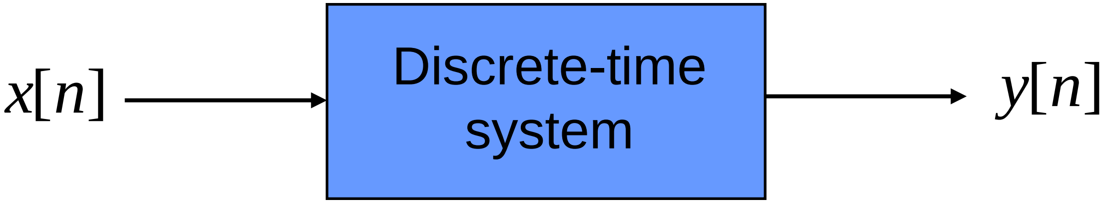
$$\Leftrightarrow u(t) = \int_{0}^{\infty} \delta(t-\sigma)d\sigma$$

✓ $x(t)\delta(t) = x(0)\delta(t)$

✓ $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$; Sampling property of the unit impulse

1.5 Continuous-Time and Discrete-Time Systems





1.5 Continuous-Time and Discrete-Time Systems

Example 1.8 - RC Circuit

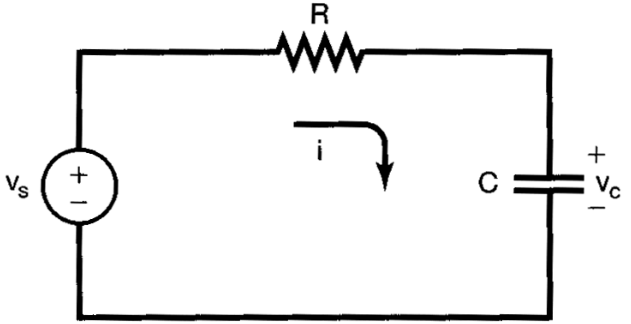


Figure 1.1 A simple *RC* circuit with source voltage v_s and capacitor voltage v_c.

$$i(t) = \frac{v_s(t) - v_c(t)}{R} \qquad i(t) = C\frac{dv_c(t)}{dt}$$

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

1.5 Continuous-Time and Discrete-Time Systems

Example 1.9 (Fig. 1.2, p. 2)

$$f(t) = m\frac{dv(t)}{dt} + \rho v(t)$$

$$\frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

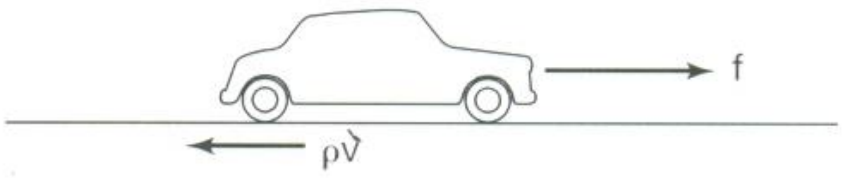


Figure 1.2 An automobile responding to an applied force f from the engine and to a retarding frictional force ρv proportional to the automobile's velocity v.

Example 1.10 - discrete system(balance in a bank account)

$$y[n] = 1.01y[n-1] + x[n]$$

$$y[n] - 1.01y[n-1] = x[n]$$

1.6 Basic System Properties

1.6.1 Systems with and without memory

Memoryless system

$$y[n] = (2x[n] - x^2[n])^2$$

$$y(t) = x(t)$$

$$y(t) = R\,x(t)$$

$$y[n] = x[n]$$

System with memory

$$y[n] = \sum_{k=-\infty}^{n} x[k] \qquad y[n] = x[n-1]$$

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n] \iff y[n] = y[n-1] + x[n]$$

1.6.2 Invertibility and inverse systems

Invertible systems (one-to-one mapping)

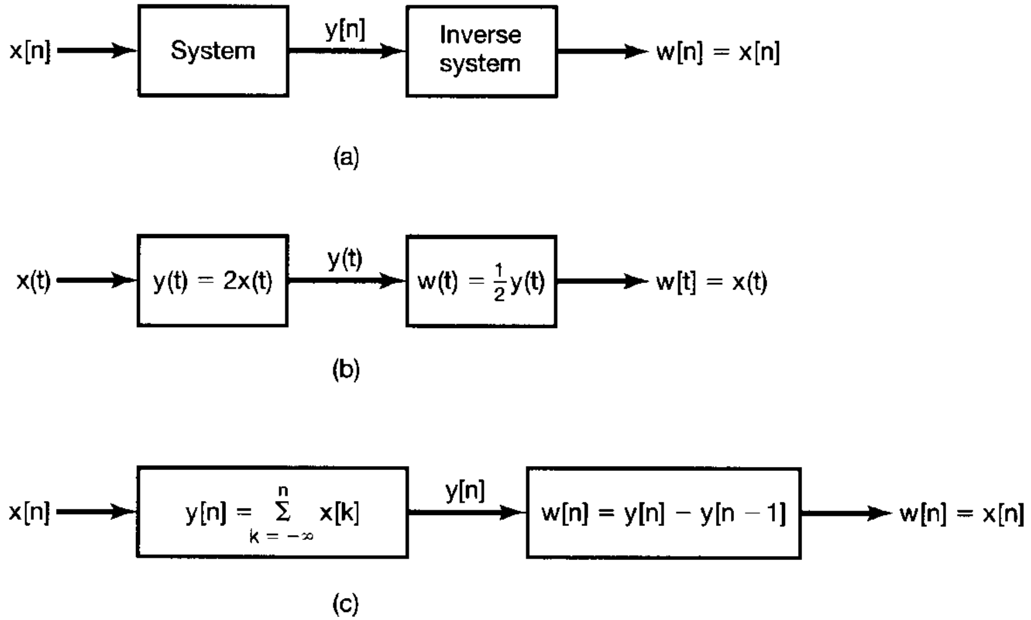


Figure 1.45 Concept of an inverse system for: (a) a general invertible system; (b) the invertible system described by eq. (1.97); (c) the invertible system defined in eq. (1.92).

Noninvertible systems : $y[n]=0 \qquad y(t)=x^2(t)$

1.6.3 Causality

Causal system

- The output at a time depends only on the input values at that time and up to that time.
- *Non-anticipative*

Non-causal system

$$y[n] = x[n] - x[n+1]$$

$$y(t) = x(t+1)$$

Example : Moving average, filtering of discrete sequences - non-causal

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]$$

Example 1.12

$y[n] = x[-n]$; non-causal

$y(t) = x(t)\cos(t+1)$; causal memoryless

1.6.4 Stability

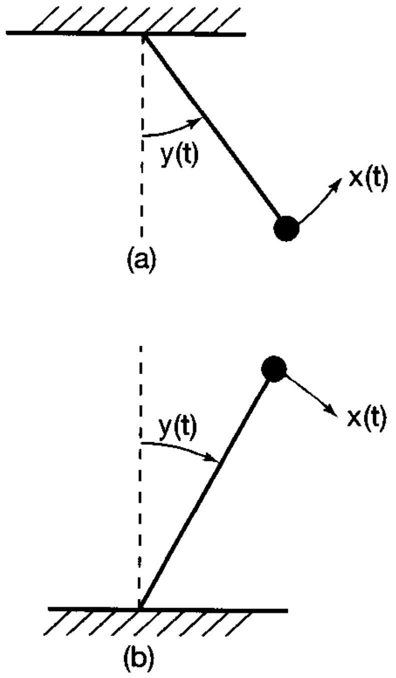


Figure 1.46 (a) A stable pendulum; (b) an unstable inverted pendulum.

Example 1.13

Stability : BIBO (bounded input bounded output) Stability

$y(t) = tx(t)$ Unstable system

$y(t) = e^{x(t)}$ Stable system

1.6.5 Time Invariance

$$\text{input} \quad x(t) \;\Leftrightarrow\; \text{output} \quad y(t)$$

$$\text{input} \quad x(t-t_0) \;\Leftrightarrow\; \text{output} \quad y(t-t_0)$$

Example 1.15 : time-varying system

$$y[n] = nx[n]$$

Example 1.16

$$y(t) = x(2t)$$

$x_1(t) \rightarrow y_1(t)$ then $y_1(t) = x_1(2t)$

Let $x_2(t) = x_1(t - t_0) \rightarrow y_2(t)$

Check $y_2(t) = y_1(t - t_0)$?

$$y_2(t) = x_2(2t) = x_1(2t - t_0)$$
$$= x_1\left(2(t - \tfrac{t_0}{2})\right)$$
$$= y_1(t - \tfrac{t_0}{2})$$
$$\neq y_1(t - t_0)$$

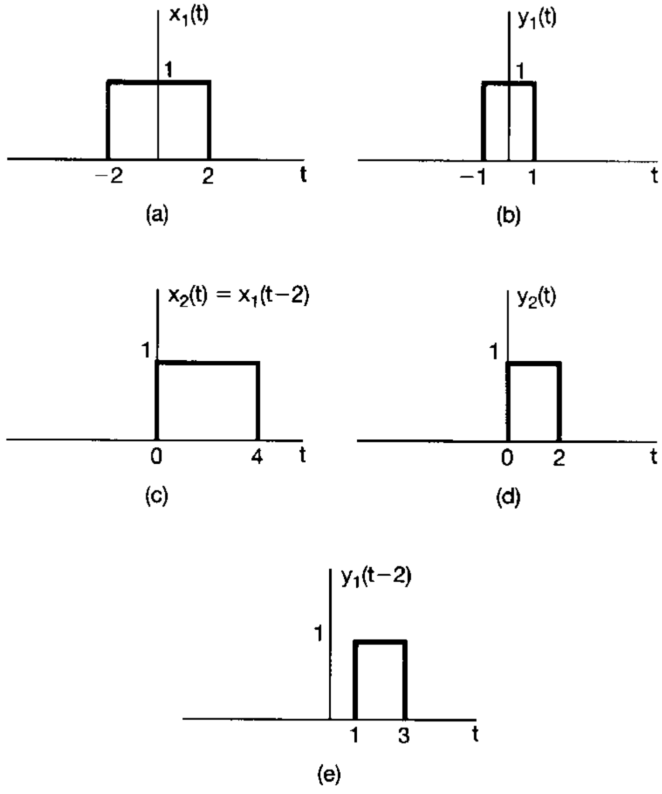


Figure 1.47 (a) The input $x_1(t)$ to the system in Example 1.16; (b) the output $y_1(t)$ corresponding to $x_1(t)$; (c) the shifted input $x_2(t) = x_1(t-2)$; (d) the output $y_2(t)$ corresponding to $x_2(t)$; (e) the shifted signal $y_1(t-2)$. Note that $y_2(t) \neq y_1(t-2)$, showing that the system is not time invariant.

1.6.6 Linearity

For any complex constants a and b,

$$\text{continuous time}: ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$$

$$\text{discrete time}: ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$$

Superposition principle (Additivity + Homegeneity)

Example 1.17 - 1.20 Is the system linear?

$y(t) = tx(t)$ linear

$y(t) = x^2(t)$ Non-linear

$y[n] = \text{Re}\{x[n]\}$ **Non-linear**

$y[n] = 2x[n] + 3$ **Non-linear**

The difference between the responses to any two inputs to an incrementally linear system is a linear function of the difference between two inputs**.**

incrementally linear system