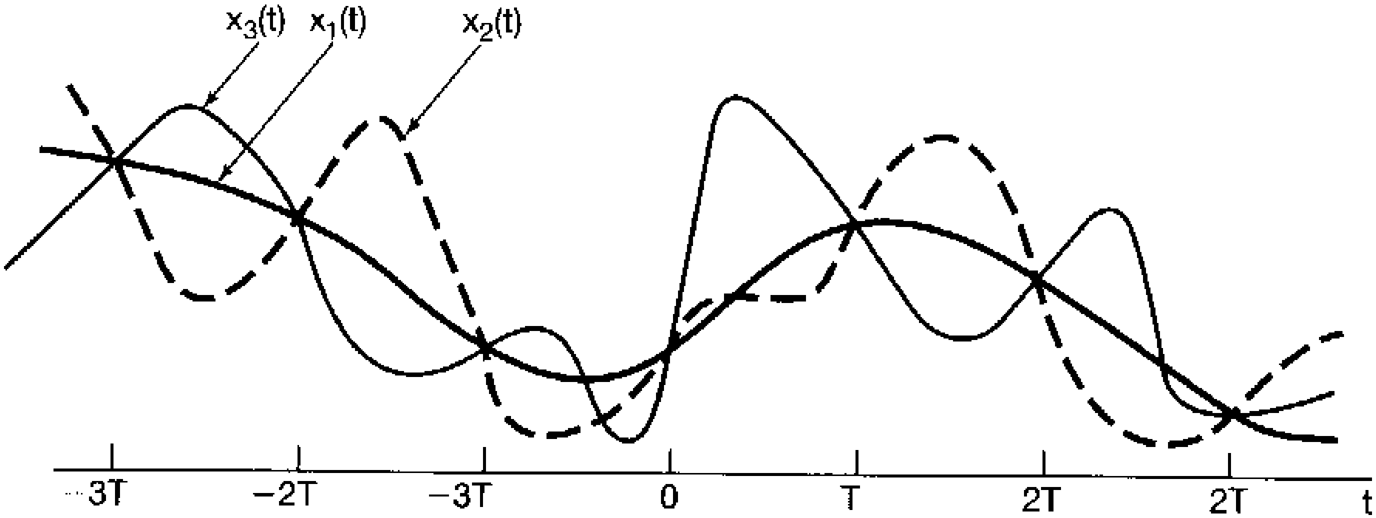
7

Sampling

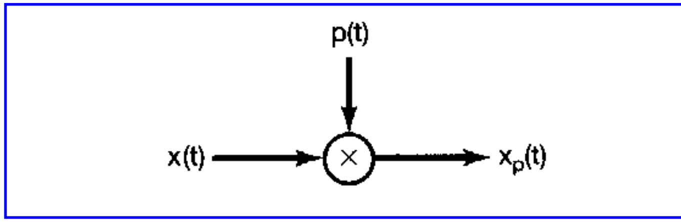
7.1 Representation of a continuous-time signal by its samples
: The sampling theorem



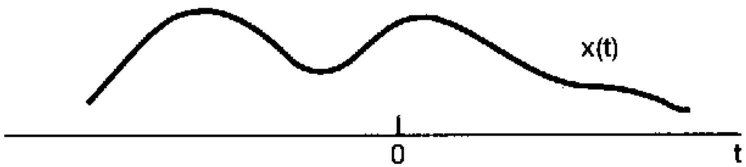
Sample → Continuous-time signal : not unique

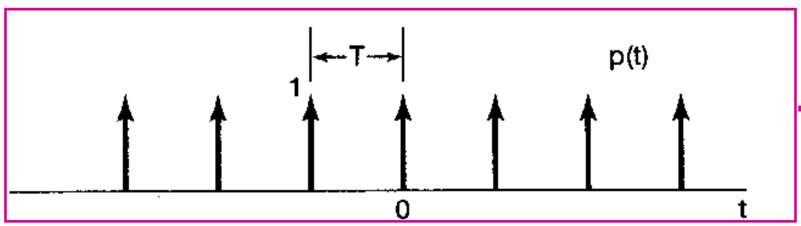
* For a Band-limited signal → unique restoration

7.1.1 Impulse-train sampling

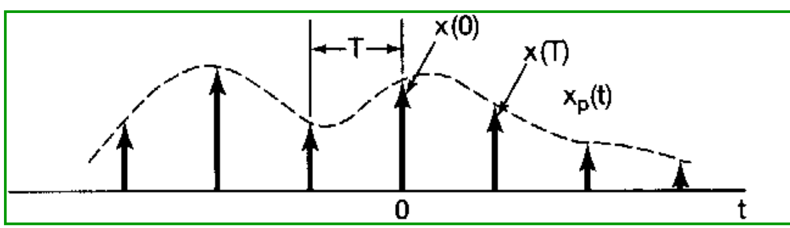


$$x_p(t) = x(t)\,p(t)$$





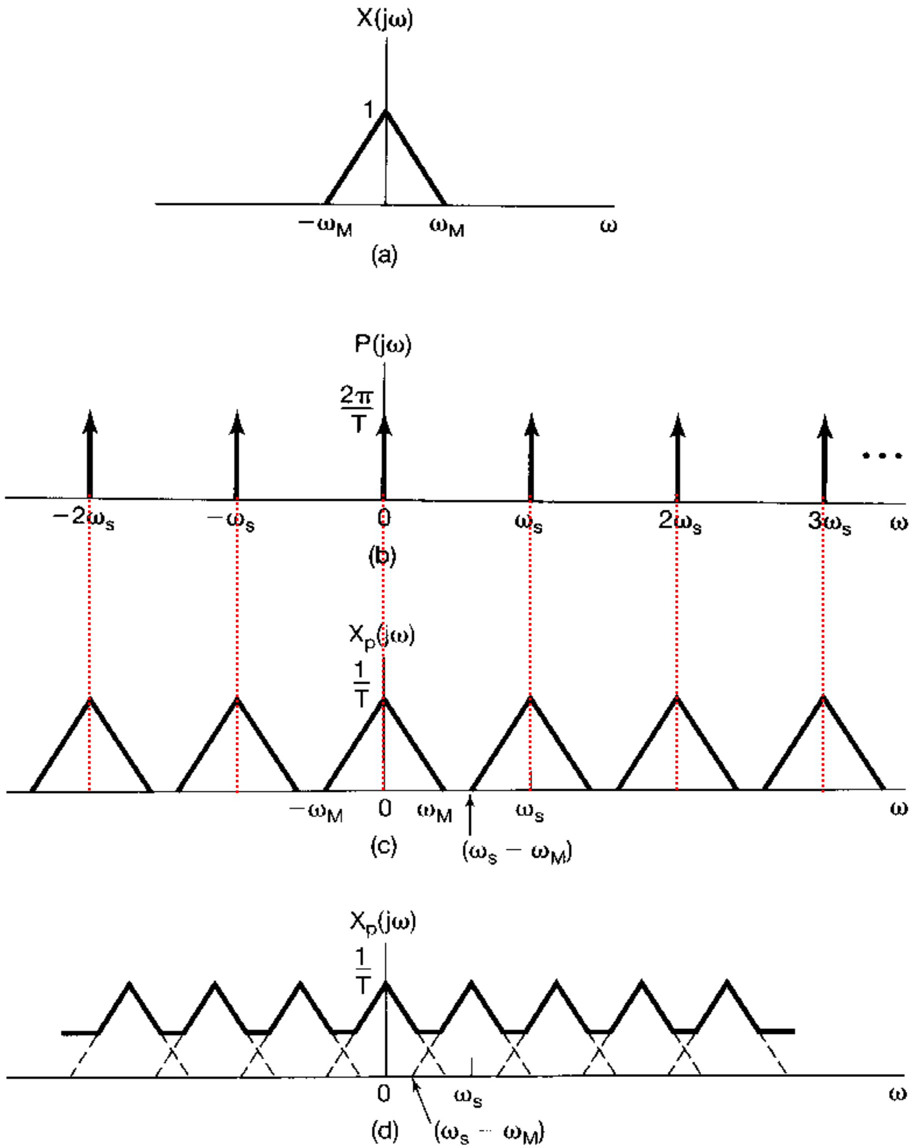
$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\,\delta(t - nT)$$

Continuous-time Fourier Transform

$$X_p(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * P(j\omega)\right]$$



$$p(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$

Continuous-time Fourier Transform

$$\omega_s=\frac{2\pi}{T}$$

$$P(j\omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta(\omega-k\omega_s)$$

$$X_p(j\omega)=\frac{1}{2\pi}\left[X(j\omega)*P(j\omega)\right]$$

$$X_p(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{+\infty}X(j(\omega-k\omega_s))$$

- Sampling Theorem:

  Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$.

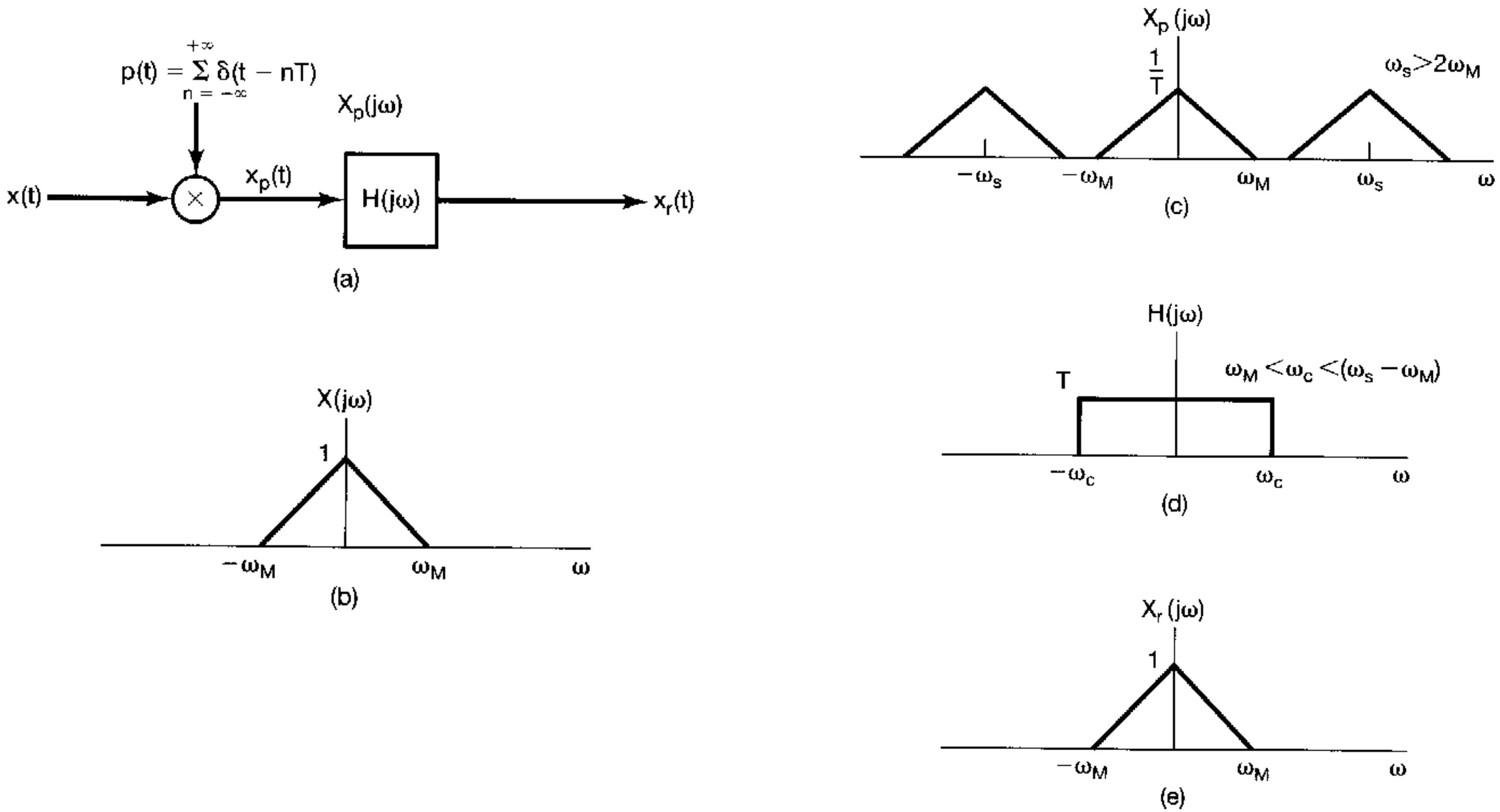
  Then, $x(t)$ is uniquely determined by its samples $x(nT), n = 0, \pm 1, \cdots,$

  if

  $$\omega_s > 2\omega_M,$$

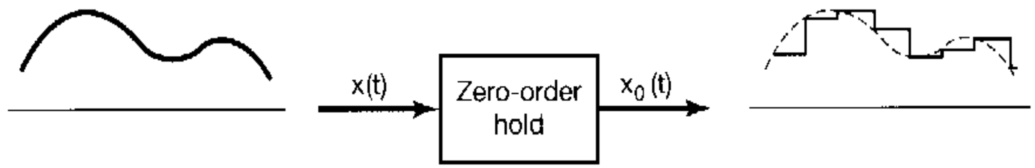
  where

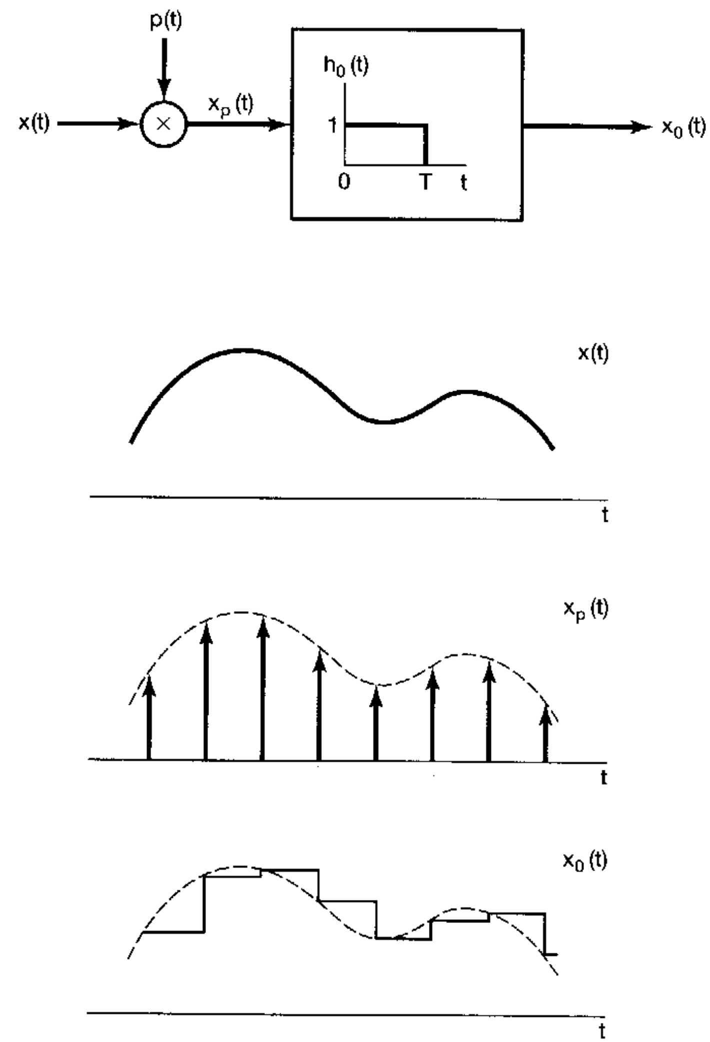
  $$\omega_s = \frac{2\pi}{T}$$



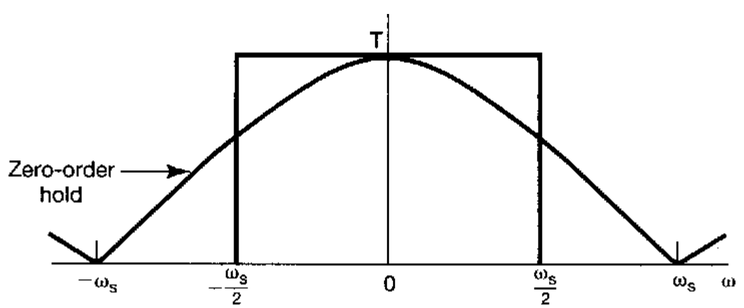
Exact recovery of a continuous-time signal from its samples

7.1.2 Sampling with a zero-order hold

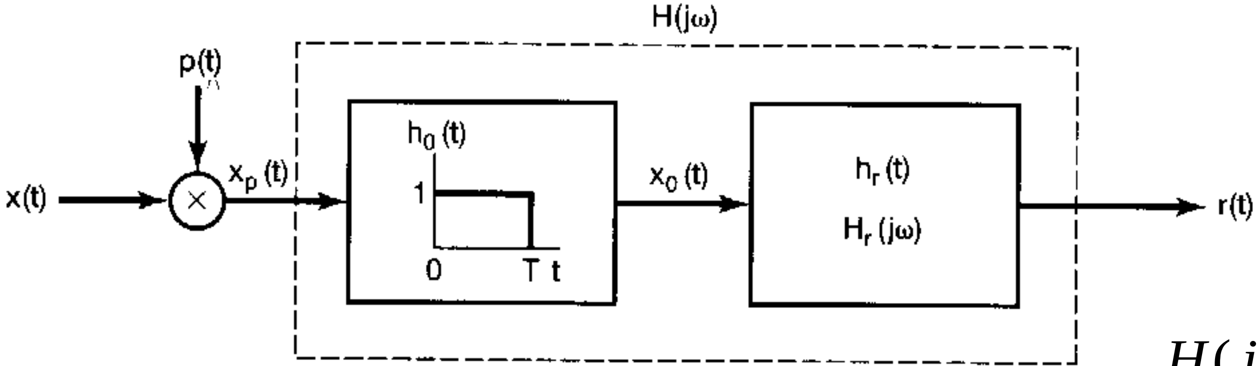




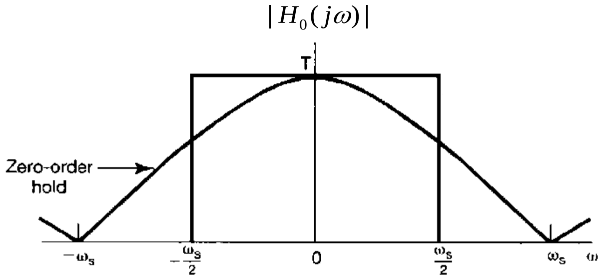
$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right]$$

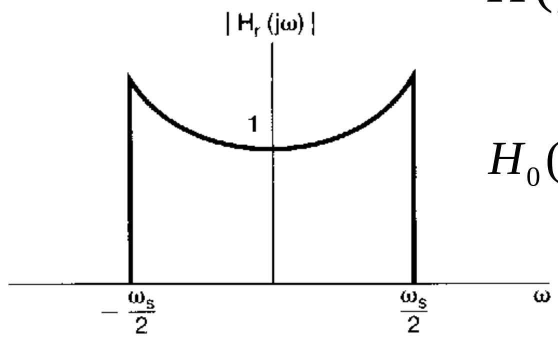


Zero-order hold with a reconstruction filter

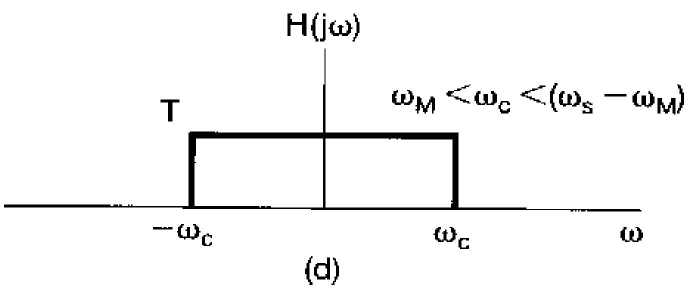


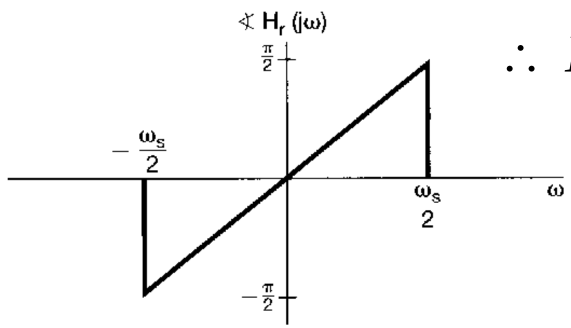
$$H(j\omega) = H_0(j\omega)H_r(j\omega)$$





$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right]$$





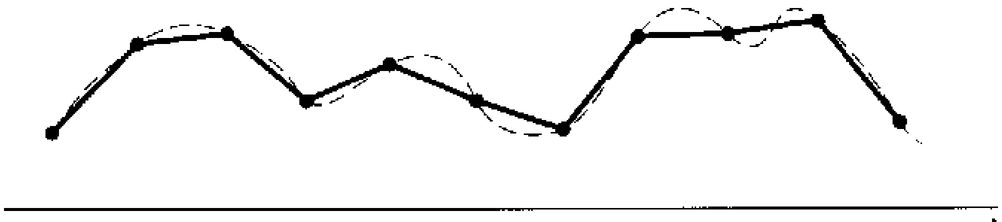
$$\therefore\ H_r(j\omega) = \frac{e^{j\omega T/2}H(j\omega)}{\frac{2\sin(\omega T/2)}{\omega}}$$

7.2 Reconstruction of a signal from its samples using interpolation

- One simple interpolation procedure: zero-order hold

The fitting of a continuous signal to a set of sample values

- Linear interpolation

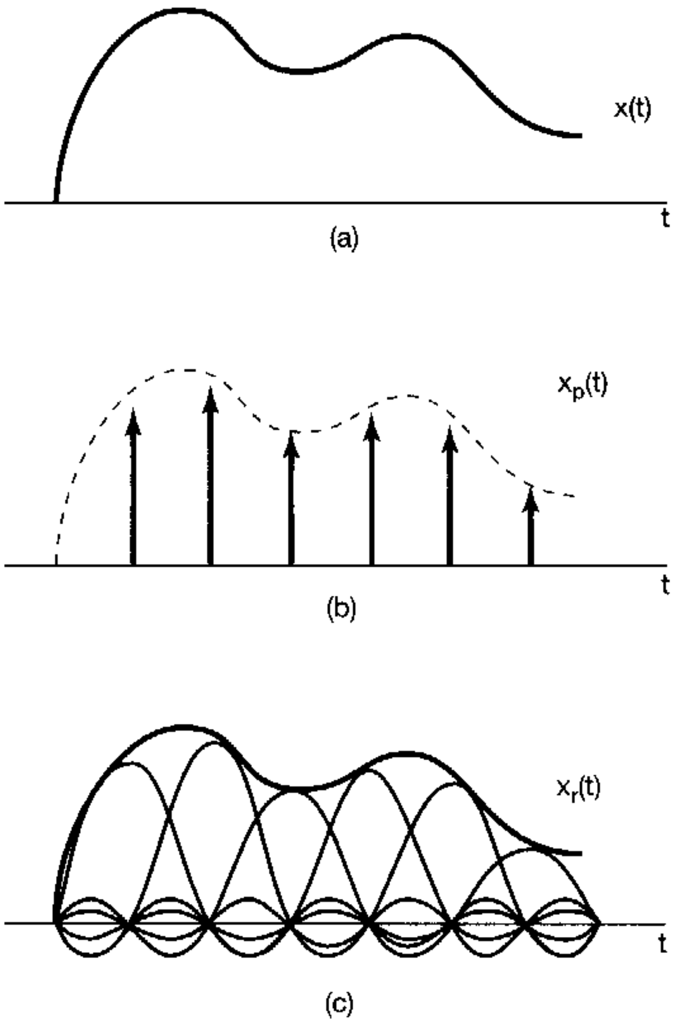


- More complicated interpolation formulas: higher order polynomials or other mathematical functions

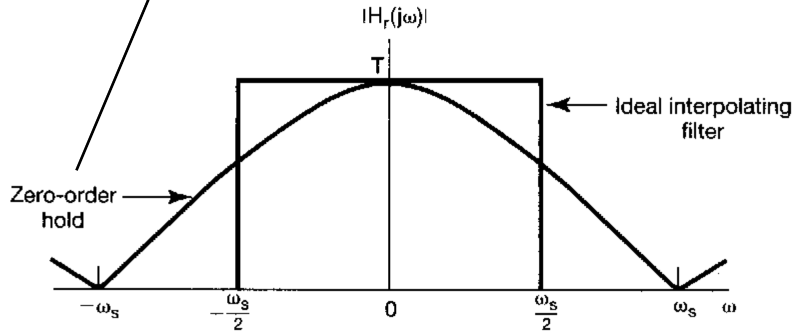
- Interpolation by ideal low-pass filtering

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t-nT) \qquad h(t) = \frac{\omega_c T \sin(\omega_c t)}{\pi \omega_c t} \qquad \left(\omega_c = \frac{\omega_s}{2}\right)$$

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{\omega_c T}{\pi} \frac{\sin(\omega_c(t-nT))}{\omega_c(t-nT)}$$



$$H_0(j\omega) = e^{-j\omega T/2}\left[\frac{2\sin(\omega T/2)}{\omega}\right]$$



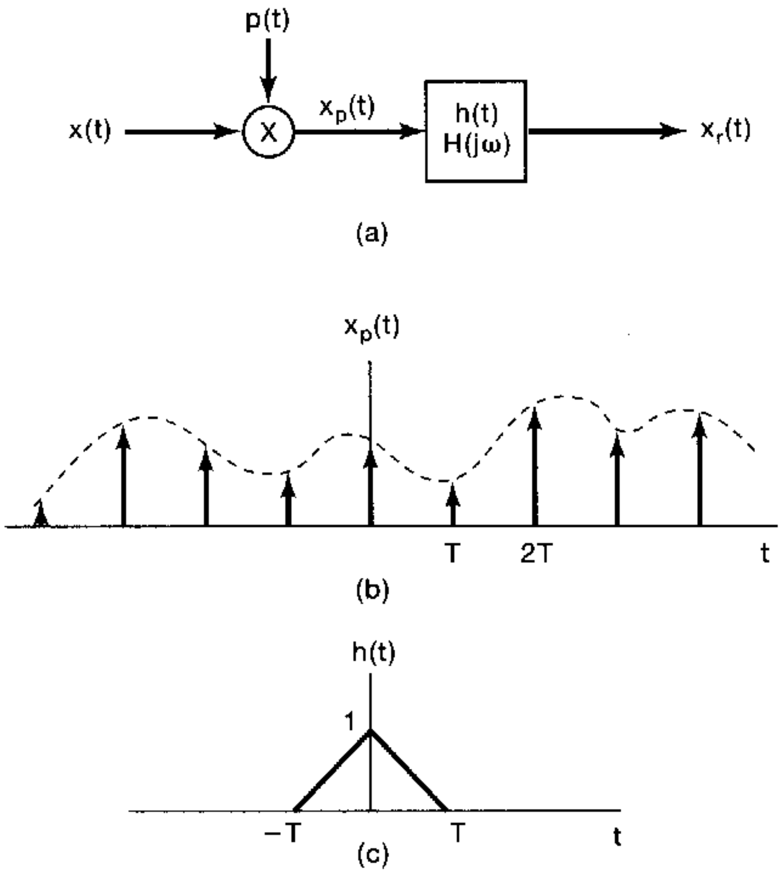
Interpolation by ideal low-pass filtering ➡ Band-limited interpolation

• Zero-order hold: discontinuous signal

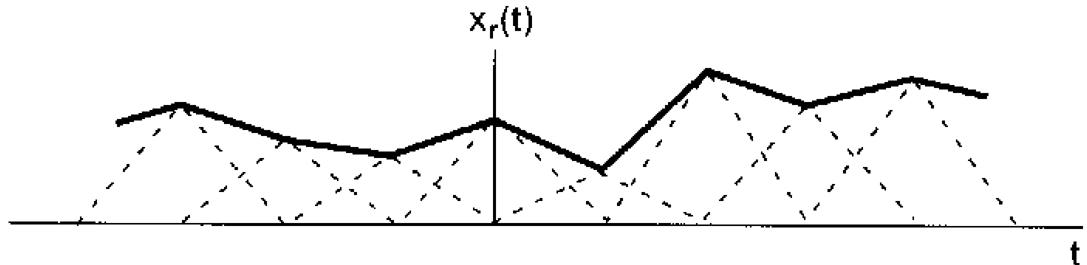
• First-order hold (linear interpolation): continuous signal, discontinuous derivative

• Second-order hold: continuous signal, continuous first derivative, discontinuous second derivative

First-order hold

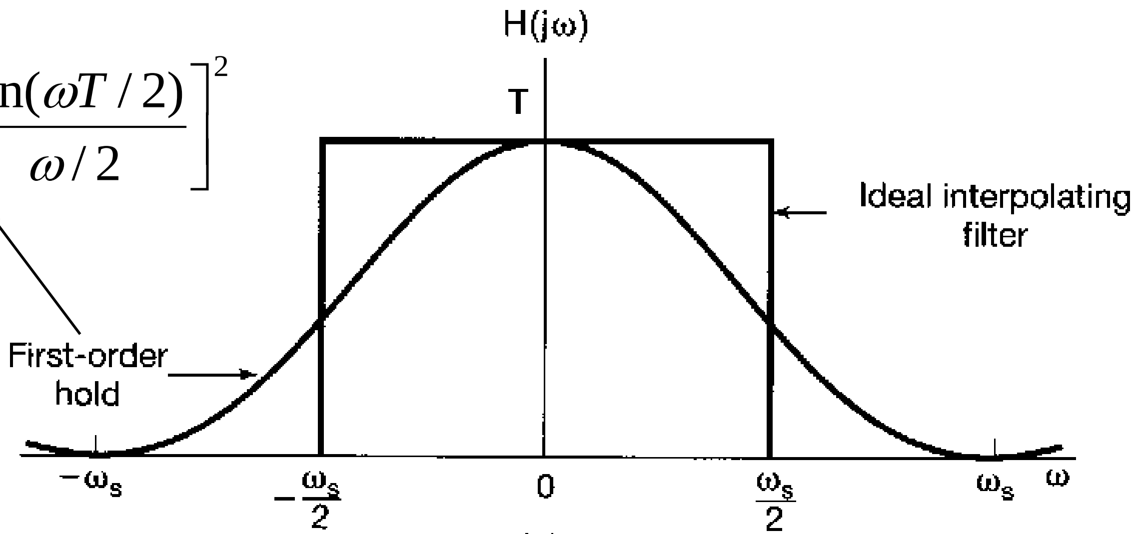


First-order hold



(d)

$$H(j\omega) = \frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$$



(e)

7.3 The effect of undersampling : aliasing $(\omega_s < 2\omega_M)$

$x(t) = \cos \omega_0 t$

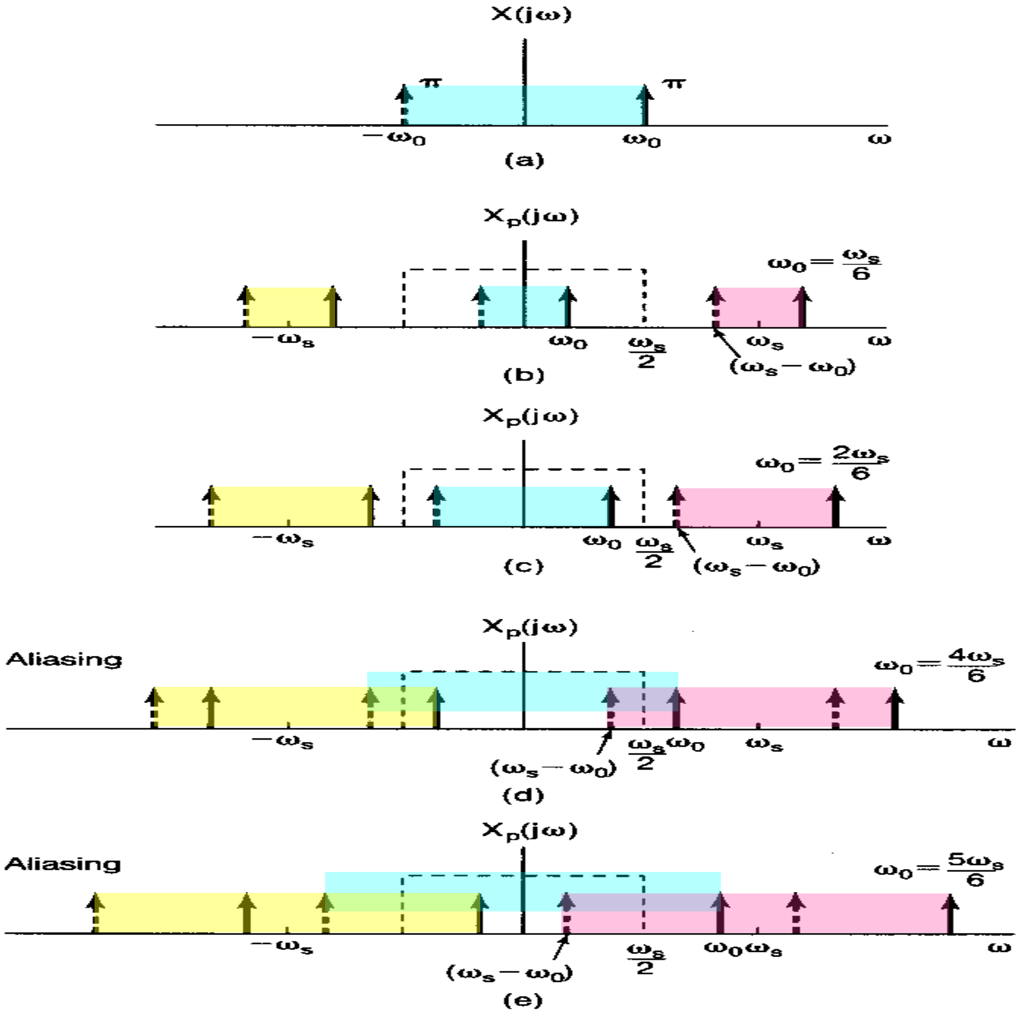
b) $\omega_0 = \omega_s / 6 \quad x_r(t) = \cos \omega_0 t$

c) $\omega_0 = 2\omega_s / 6 \quad x_r(t) = \cos \omega_0 t$

d) $\omega_0 = 4\omega_s / 6 \quad x_r(t) = \cos(\omega_s - \omega_0)t$

e) $\omega_0 = 5\omega_s / 6 \quad x_r(t) = \cos(\omega_s - \omega_0)t$

f) $\omega_0 = \omega_s \quad x_r(t) = 1$

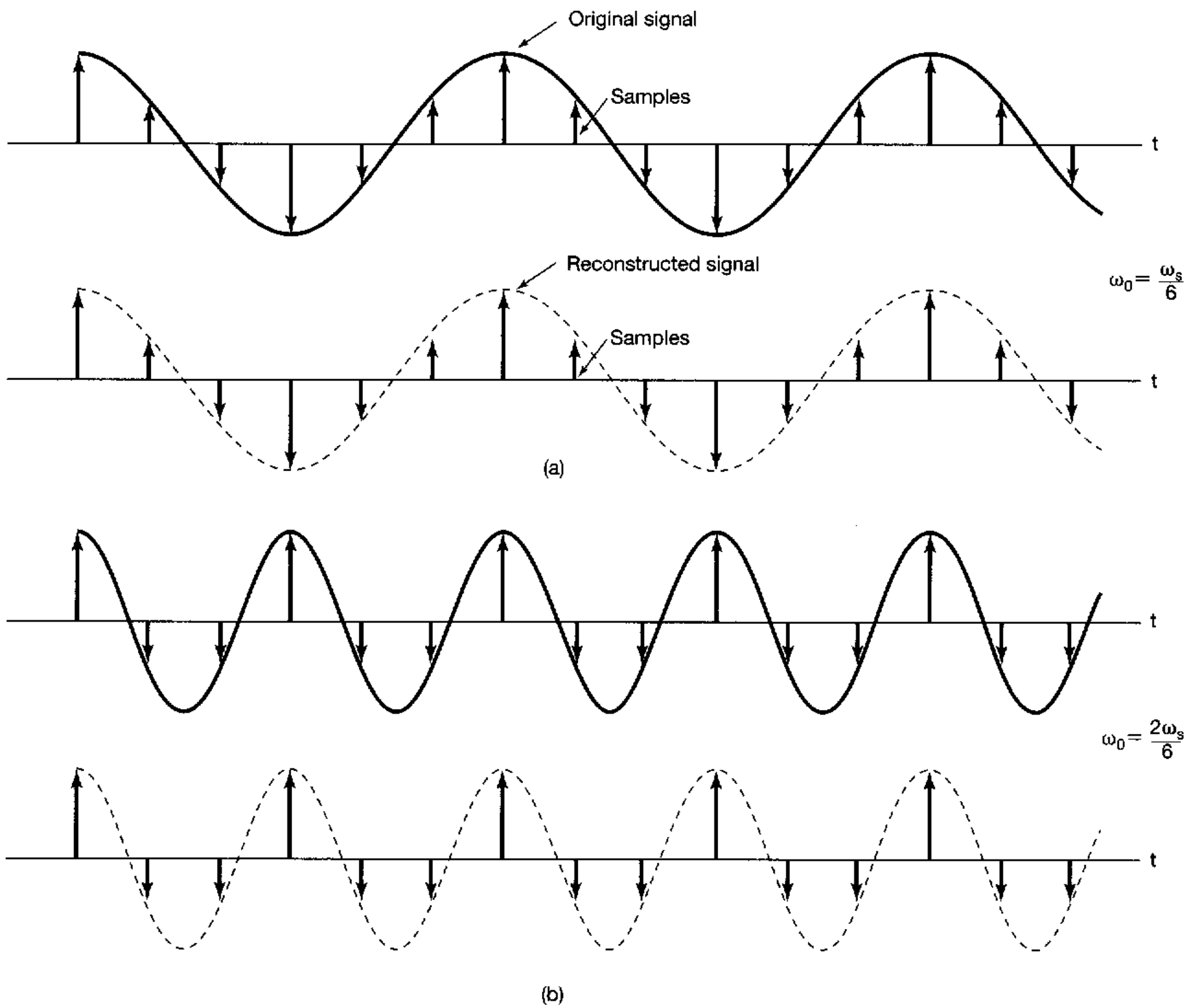


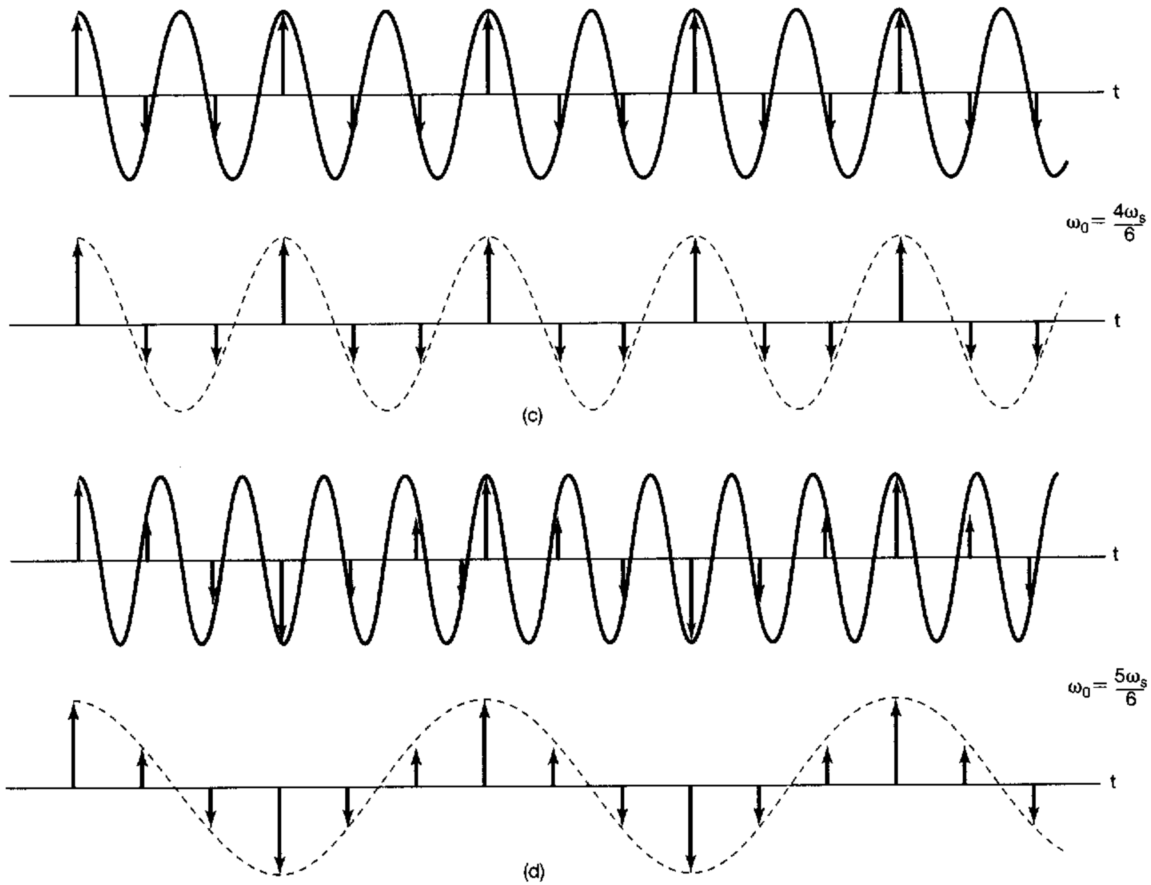
If $x(t) = \cos(\omega_0 t + \phi)$

- amplitude of solid line : $\pi e^{j\phi}$
- amplitude of dashed line : $\pi e^{-j\phi}$

if $\omega_0 = 4\omega_s / 6$,

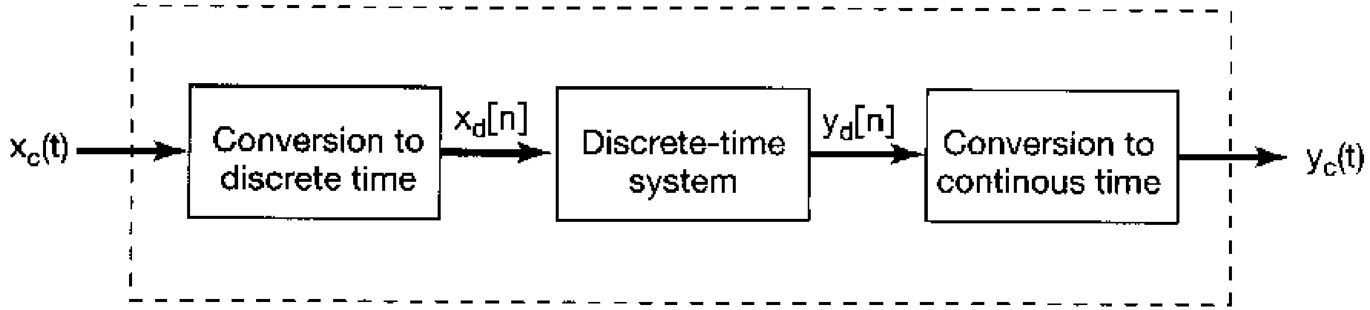
then $x_r(t) = \cos((\omega_s - \omega_0)t - \phi)$


Original signal
Samples
t
Reconstructed signal
$\omega_0 = \frac{\omega_s}{6}$
Samples
t
(a)
t
$\omega_0 = \frac{2\omega_s}{6}$
t
(b)

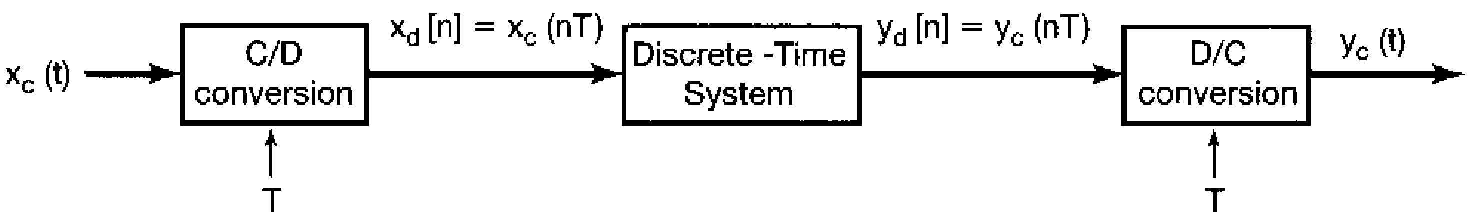


- Examples of the effect of undersampling (aliasing)
  - ➢ Strobe effect (p. 533, Fig. 7.18)
  - ➢ The wheels of a stagecoach in Western movies
- Note) Irrespective of aliasing, $x_r(nT) = x(nT), \quad n = 0, \pm 1, \pm 2, \ldots$

7.4 Discrete-time processing of continuous-time signals

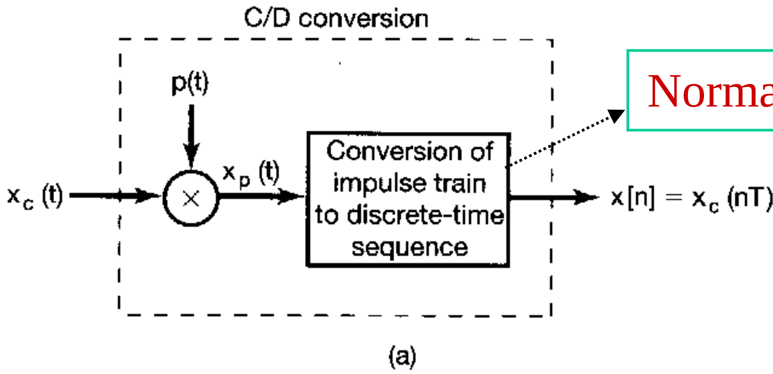


$$x_d[n] = x_c(nT) \qquad y_d[n] = y_c(nT)$$



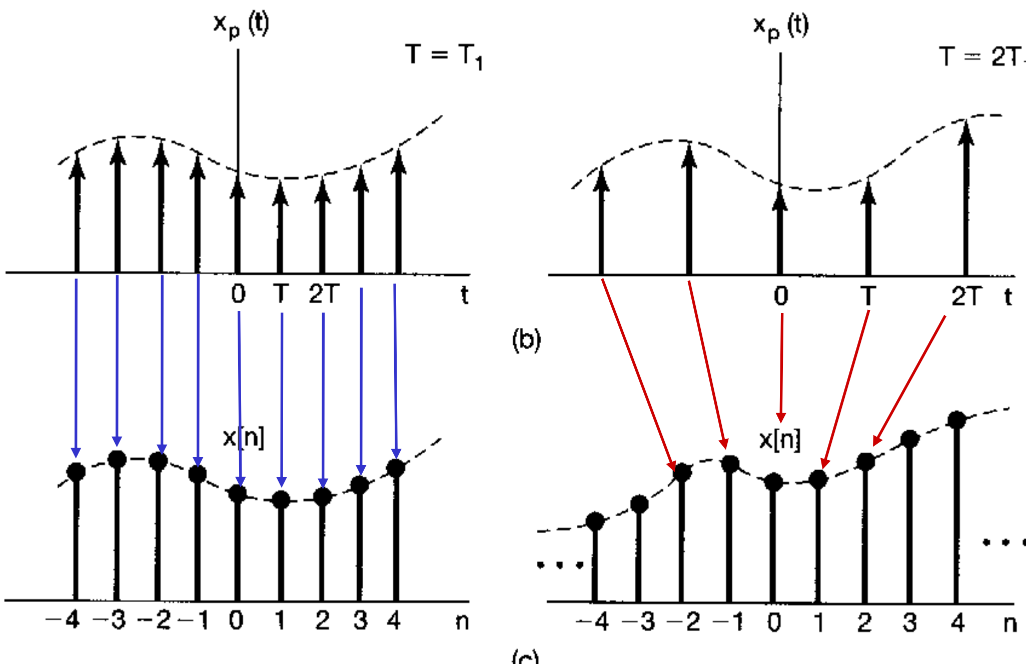
< Notation for continuous-to-discrete-time conversion and discrete-to-continuous-time conversion>

Sampling with a periodic impulse train followed by conversion to a discrete-time sequence



Normalization in time (scaling in time)

cf.) Fig. 1.12 at p.9



ω : continuous-time frequency variable

Ω : discrete-time frequency variable

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t-nT) \xrightarrow{\text{CTFT}} X_p(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\omega nT}$$

$x_d[n]$

DTFT

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega n}$$

Compare

$\Omega = \omega T$

$$X_d(e^{j\Omega}) = X_p(j\Omega / T)$$

Since $X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$,

$$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\Omega - 2\pi k)/T)$$

$$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\Omega - 2\pi k)/T)$$