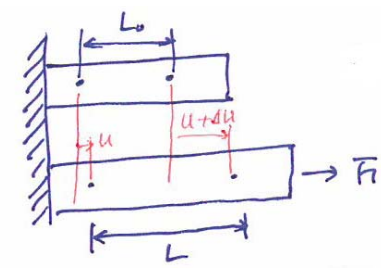
3장 변형률

3.1 변위와 무한소변형률



표점거리 : $L_0 \rightarrow L$

변형(deformation

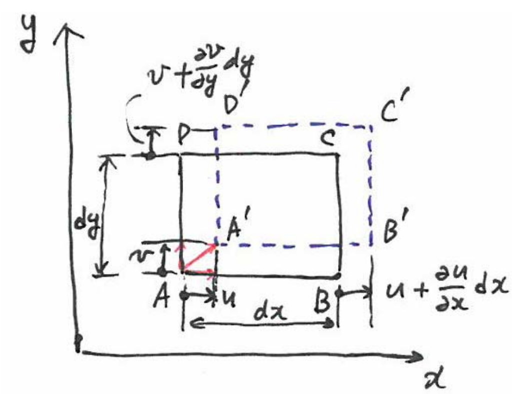
cf. 변위(displacement)

$$e = \frac{L - L_0}{L_0} = \frac{\Delta L}{L_0} \qquad (\ e\ :\ \text{평균선형변형률}\ )$$

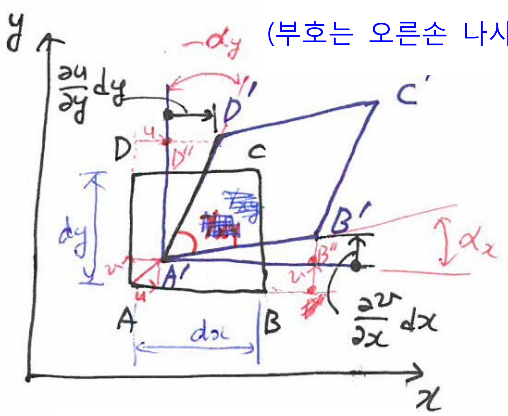
- 한 점에서의 변형률: 표점거리가 0에 접근할 때 변형과 표점거리와의 비

$$\epsilon_x = \lim_{L \to 0} \frac{\Delta u}{L} = \frac{du}{dL} \qquad \text{—— ①}$$

- 평면 변형률(plane strain) ; where in all points in the body,
  before and after application of load, remain in the same plane



(normal strain)



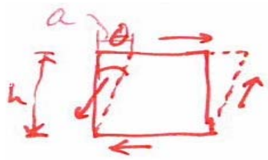
(부호는 오른손 나사 법칙)

(shearing strain)

$$\epsilon_x = \frac{\partial u}{\partial x}\ ,\ \epsilon_y = \frac{\partial v}{\partial y}$$

공칭전단변형률 - 각의 변화량으로 나타냄

$\gamma = \dfrac{a}{h} = \tan\theta \approx \theta$



- Now consider the change of angle DAB (in shearing strain)

  ○ Assume the angle α_x : between AB and $A'B'$

  $\rightarrow$ approximation $\alpha_x \approx \tan\alpha_x (= \dfrac{\partial v}{\partial x})$

  ○ Also, in view of the smallness of α_x , $\rightarrow$ the normal strain is very small

  $\therefore AB \approx A'B''$

  ○ As $\alpha_x = \dfrac{\partial v}{\partial x}$ $(\because \tan\alpha_x \approx \alpha_x = \dfrac{B''B}{A'B'} \simeq \dfrac{B''B'}{AB} = \dfrac{\frac{\partial v}{\partial x}dx}{dx} = \dfrac{\partial v}{\partial x})$

  (counterclockwise rotation : +)

  $\rightarrow \; -\alpha_y = \dfrac{\partial u}{\partial y}$

공칭전단변형률, 텐서량은 이것, 각도 변화량만 나타냄

- The total angular change of angle DAB, γ_{xy}

$$\gamma_{xy} = \alpha_x - \alpha_y = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \text{—— ②}$$

  $\Rightarrow$ The shear strain is positive when the angle between two positive axes decrease.

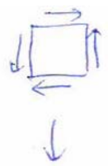
- In the case of 3D element,

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \epsilon_z = \frac{\partial w}{\partial z}$$

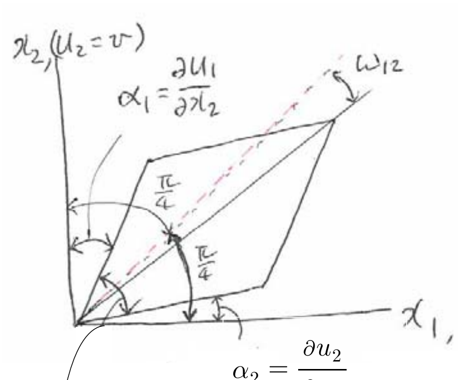
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \; \text{(공칭전단변형률)} \qquad \text{—— ③}$$

$$\rightarrow \gamma_{xy} = \gamma_{yx}, \quad \gamma_{yz} = \gamma_{zy}, \quad \gamma_{zx} = \gamma_{xz} \qquad \text{—— ④}$$

- Eq ③ and ④ called to the strain-displacement relations of continuum mechanics



- 순수전단변형만을 고려해 보자

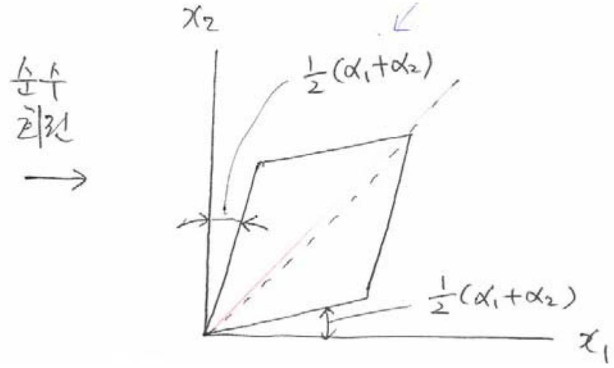


$\alpha_2 = \dfrac{\partial u_2}{\partial x_1}$

$\dfrac{\pi}{2} - (\alpha_1 + \alpha_2) = \gamma_{12}$

< 변형 후의 상태>

전단변형률



공칭전단변형률 γ_{12}는 회전(순수회전)에 영향을 받지 않음

< 순수전단 상태>

- 따라서 전단변형률 $\epsilon_{12}(tensor) = \dfrac{1}{2}(\alpha_1 + \alpha_2) = \dfrac{1}{2}(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}) = \dfrac{1}{2}\gamma_{12}$ 가 되고

  $\epsilon_{21} = \epsilon_{12}$ 가 된다.

- 그러나 변형 후의 상태가 되기 위해서는 순수전단 상태를 ω_{12} 만큼 회전시켜야 한다.

$$
\begin{aligned}
\omega_{12} &= \frac{\pi}{4} - [\frac{1}{2}(\frac{\pi}{2} - (\alpha_1 + \alpha_2)) + \alpha_2] \\
&= \frac{\pi}{4} - [\frac{\pi}{4} - \frac{\alpha_1 + \alpha_2}{2} + \alpha_2] \\
&= \frac{1}{2}(\alpha_1 - \alpha_2) = \frac{1}{2}(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1})
\end{aligned}
$$

- 일반화된 형태 $\quad \omega_{ij} = \dfrac{1}{2}(\dfrac{\partial u_i}{\partial x_j} - \dfrac{\partial u_j}{\partial x_i})$: 회전텐서

- Let $e_{11} = \dfrac{\partial u_1}{\partial x_1}, e_{12} = \dfrac{\partial u_1}{\partial x_2},\ e_{ij} = \dfrac{\partial u_i}{\partial x_j}$ 로 두면,

$$
\begin{aligned}
e_{ij} &= \frac{1}{2}(e_{ij} + e_{ji}) + \frac{1}{2}(e_{ij} - e_{ji}) \\
&= \epsilon_{ij} + \omega_{ij}
\end{aligned}
$$

(e_{ij}: 상대변위텐서, ϵ_{ij}: 변형률텐서, ω_{ij}: 회전텐서)

$$e_{ij}(\text{상대변위텐서})=\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

→ 변위 u_i는, $u_i = e_{ij}x_j$로 나타낸다.

$$\begin{aligned} u_1 &= e_{11}x_1 + e_{12}x_2 + e_{13}x_3 \\ u_2 &= e_{21}x_1 + e_{22}x_2 + e_{23}x_3 \\ u_3 &= e_{31}x_1 + e_{32}x_2 + e_{33}x_3 \end{aligned}$$

- $e_{ij} = \epsilon_{ij} + \omega_{ij}$ 이므로
  $u_i = \epsilon_{ij}x_j + \omega_{ij}x_j$ 가 됨

here,

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$(\epsilon_{ij} = \epsilon_{ji})$

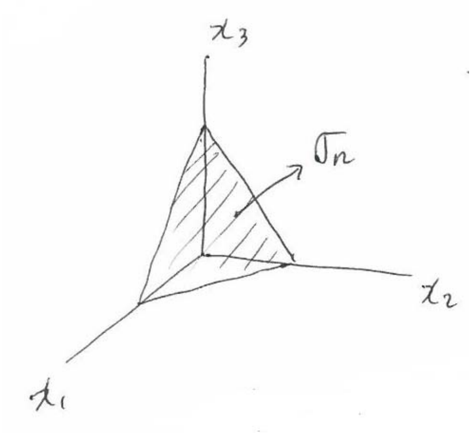
$$\omega_{ij} = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}) & 0 & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}) & 0 \end{bmatrix}$$

$(\omega_{ij} = -\omega_{ji})$

- ϵ_{ij}는 텐서량이므로 이미 설명한 응력텐서와 같은 성질을 가짐.

→ 응력에 대해 전개한 식에서 σ_{ij} 대신 ϵ_{ij}를 대입하면 바로 성립

ex)



경사면 수직응력, $\widetilde{\sigma_n}$

$$\widetilde{\sigma_n} = \sigma_{11} l_1^2 + \sigma_{22} l_2^2 + \sigma_{33} l_3^2 + 2\sigma_{12} l_1 l_2 + 2\sigma_{23} l_2 l_3 + 2\sigma_{31} l_3 l_1)\,'$$

↓ 경사면 수직면에서 수직 변형률?

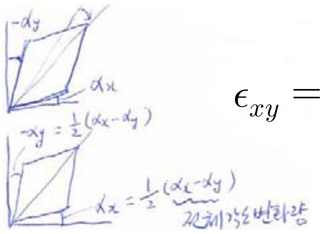
$$\widetilde{\epsilon_n} = \epsilon_{11} l_1^2 + \epsilon_{22} l_2^2 + \epsilon_{33} l_3^2 + \underbrace{2\epsilon_{12}}_{=\gamma_{12}} l_1 l_2 + \underbrace{2\epsilon_{23}}_{=\gamma_{23}} l_2 l_3 + \underbrace{2\epsilon_{31}}_{=\gamma_{31}} l_3 l_1)$$

- A succinct statement of strain

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\ i,j = x,y,z$$

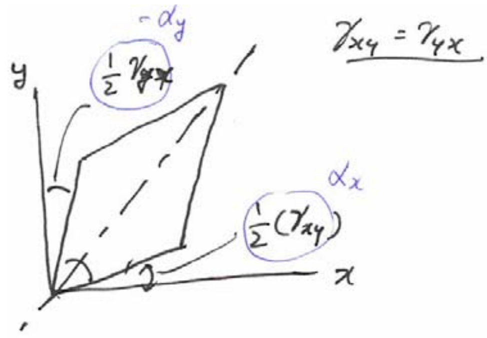
where $u_x = u,\ u_y = v,\quad x_x = x,\ x_y = y\ \ \mathrm{and}\, so\ on$

※전단변형률은 회전에 영향을 받지 않는다



ex) $i = x, j = y$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$



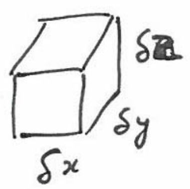
(ϵ_{xy} : 변형률텐서, 응력텐서와 같은 성질, γ_{xy} : 전체 각도변화량, 공친 전단변형률)

$$\therefore\ \epsilon_{xy} = \frac{1}{2}\gamma_{xy},\ \epsilon_{yz} = \frac{1}{2}\gamma_{yz},\ \epsilon_{zx} = \frac{1}{2}\gamma_{zx} \qquad \text{—— ⑤}$$

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \epsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \epsilon_z \end{bmatrix}$$

9개 → 6개 —— ⑥

- 체적 변형률, △(delta), dilatation
  : 체적 증가를 최초의 체적으로 나눈 값



○ 최초의 체적 : $\delta x \delta y \delta z$

○ 변형후 체적(신장 또는 압축에만 기인한다고 가정)

$$: \ \delta x(1+\epsilon_x)\delta y(1+\epsilon_y)\delta z(1+\epsilon_z)$$

$$= \delta x\delta y\delta z(1+\epsilon_x+\epsilon_y+\epsilon_z+\epsilon_x\epsilon_y+\epsilon_x\epsilon_z+\epsilon_y\epsilon_z+\epsilon_x\epsilon_y\epsilon_z)$$

$$\therefore\ \delta V = \delta x\delta y\delta z(\epsilon_x+\epsilon_y+\epsilon_z)$$

$$\triangle = \frac{\delta V}{V} = \frac{1}{\delta x \delta y \delta z}(\delta x \delta y \delta z(\epsilon_x + \epsilon_y + \epsilon_z))$$

$$= \epsilon_x + \epsilon_y + \epsilon_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \qquad —— ⑦$$

⁎ 변형의 적합조건식(strain compatibility equation)

: 6개의 변형률 성분은 3개의 변위 성분과 관계가 있기 때문에 변형률 성분 사이에 어떤 관계가 존재하지 않으면 안 된다.
이것을 '적합조건식'이라 한다.

⇒ Physically, this means that the body must be pieced together;
no voides are created in the deformed body

- In two-dimensional strain, we now proceed to develop the equations of compatibility, which establish the geometrically possible form of variation of strains from point to point within a body.

- Differentiation of ϵ_x twice with respect to y,
ϵ_y twice with respect to x,
and γ_{xy} with respect to x and y

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^3 u}{\partial x \partial y^2},$$

$$\frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y},$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

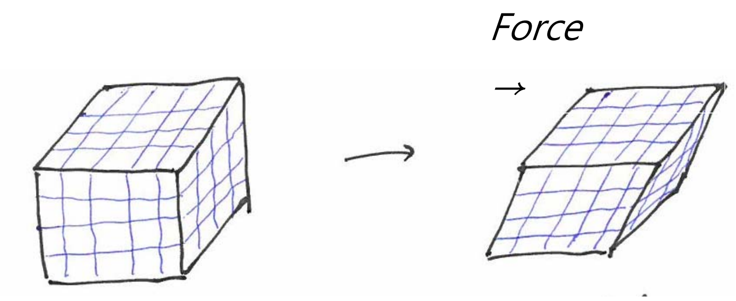
$$\therefore \ \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad \text{—— ⑧}$$

변위: 3개 $(u,\ v,\ w)$
변형률: 6개 $(\epsilon_x, \epsilon_y,\ \epsilon_z,\ \gamma_{xy},\ \gamma_{xz},\ \gamma_{zy})$ (※ 변형률 성분들은 서로 독립적일 수가 없다)

- In three-dimensional equation of compatibility, they can be obtained by the same manner.

$$\left.\begin{aligned}
&\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y},\quad 2\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\
&\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z},\quad 2\frac{\partial^2 \epsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\
&\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{xz}}{\partial z \partial x},\quad 2\frac{\partial^2 \epsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)
\end{aligned}\right] \quad ⑨$$

→ These equations were firstly derived by Saint-Venant in 1860.



Prior to deformation *after deformation*

⇒ 변형률 적합조건식이 만족하지 않으면 위와 같은 규칙적인
변형이 발생되지 못함.