Chapter 9 Vector Differential Calculus, Grad, Div, Curl

9.1 Vectors in 2-Space and 3-Space

9.2 Inner Product (Dot Product)

9.3 Vector Product (Cross Product, Outer Product)

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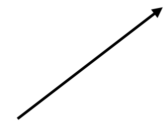
Chapter 9 Vector Differential Calculus, Grad, Div, Curl

9.1 Vectors in 2-Space and 3-Space

Two kinds of quantities used in physics, engineering and so on.

A *scalar* : A quantity representing magnitude.
A *vector* : A quantity representing magnitude and direction.



A vector is represented by an arrow.
The *tail* is called *initial point.*
The *head* (or the tip) is called *terminal point*

The distance between the initial and the terminal points is called the *distance*, *magnitude*, or *norm*.

A velocity is a vector, $\vec{v}$, and its norm is $|\vec{v}|$.

A vector of length 1 is called a *unit vector*.

Definition Equality of Vectors

> Two vectors $\vec{a}$ and $\vec{b}$ are equal, $\vec{a}=\vec{b}$, if they have the same length and the same direction.
> A translation does not change a vector.

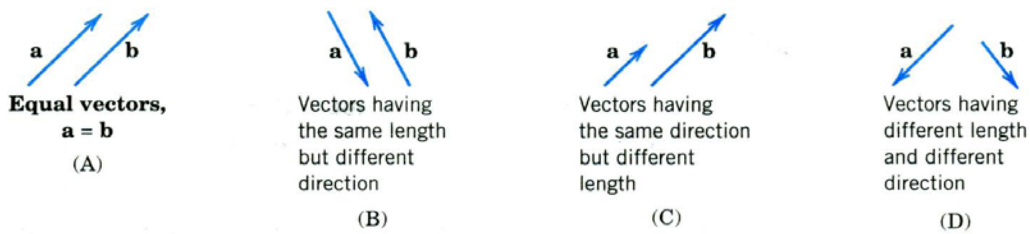


Fig. 164. (A) Equal vectors. (B)–(D) Different vectors

Components of a Vector

A vector $\vec{a}$ is given with initial point $P:(x_1, y_1, z_1)$ and terminal point $Q:(x_2, y_2, z_2)$.

The vector $\vec{a}$ has three *components* along x, y, and z coordinates.

$$\vec{a}=[a_1,\ a_2,\ a_3]$$

where

$$a_1 = x_2 - x_1, \qquad a_2 = y_2 - y_1, \qquad a_3 = z_2 - z_1$$

The length $|\vec{a}|$ is given by

$$\boxed{|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$$

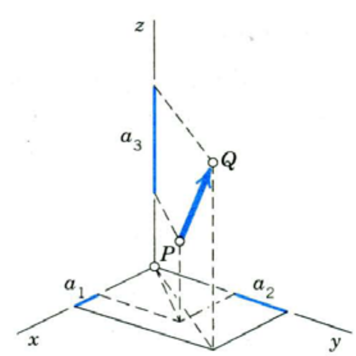


Fig. 166. Components of a vector

Example 1 Components and Length of a Vector

A vector $\vec{a}$ with initial point $P:(4,\ 0,\ 2)$ and terminal point $Q:(6,\ -1,\ 2)$.

Its components are

$$a_1 = 6-4 \Rightarrow 2, \quad a_2 = -1-0 \Rightarrow -1, \quad a_3 = 2-2 \Rightarrow 0$$

Hence $\vec{a} = [2,\ -1,\ 0]$

Its length is $|\vec{a}| = \sqrt{2^2 + (-1)^2 + 0^2} \Rightarrow \sqrt{5}$

- Position vector

  A point A:(x, y, z) is given in Cartesian coordinate system.

  The *position vector* of the point A is a vector drawn from the origin to the point A.

  $$\vec{r} = [x, y, z]$$

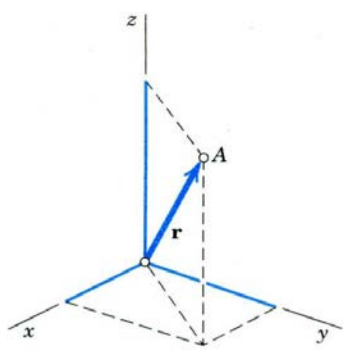


Fig. 167. Position vector **r** of a point A: (x, y, z)

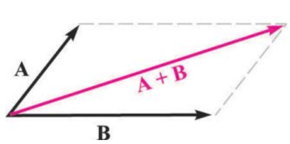
Theorem 1 Vectors as Ordered Real Triple Numbers

> A vector in a Cartesian coordinate system can be represented by three real numbers $(a_1,\ a_2,\ a_3)$, which corresponds to three components.
> Hence $\vec{a} = \vec{b}$ means that $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

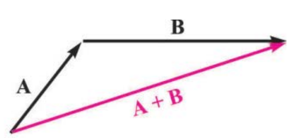
Vector Addition, Scalar Multiplication

Definition

> *Addition* of two vectors $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$
>
> $$\boxed{\vec{a} + \vec{b} = [a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3]}$$



Parallelogram rule



Head-To-Tail rule

Basic Properties of Vector Addition

$\vec{a}+\vec{b}=\vec{b}+\vec{a}$: commutative

$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$: associative

$\vec{a}+0=0+\vec{a}=\vec{a}$

$\vec{a}+(-\vec{a})=0$

A vector $-\vec{a}$ has the same length $|\vec{a}|$ as $\vec{a}$ but with the opposite direction.

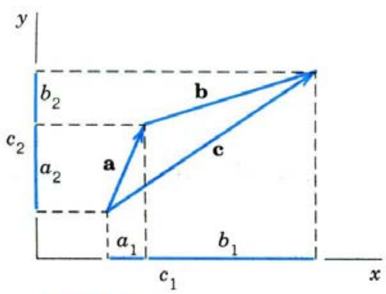


Fig. 170. Vector addition

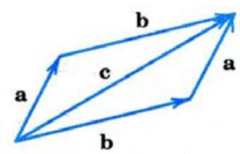


Fig. 171. Commutativity of vector addition

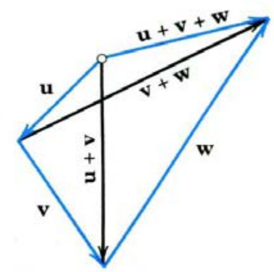


Fig. 172. Associativity of vector addition

Definition

In *scalar multiplication*, a vector $\vec{a}$ is multiplied by a scalar *c*.

$\boxed{c\vec{a}=[ca_1,\, ca_2,\, ca_3]}$

$c\vec{a}$ has the same direction with $\vec{a}$ with increased length for $c>0$,
" opposite direction " " for $c<0$

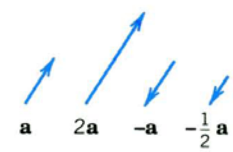


Fig. 173. Scalar multiplication

Basic Properties of Scalar Multiplication

$c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$

$(c+k)\vec{a}=c\vec{a}+k\vec{a}$

$c(k\vec{a})=(ck)\vec{a}=ck\vec{a}$

$1\vec{a}=\vec{a}$, $0\vec{a}=0$, $(-1)\vec{a}=-\vec{a}$ $\rightarrow$ $\vec{b}+(-\vec{a})=\vec{b}-\vec{a}$

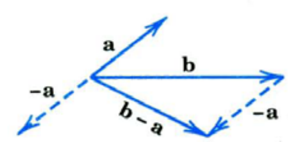


Fig. 174. Difference of vectors

Unit Vectors $\hat{i}, \hat{j}, \hat{k}$

A vector can be represented as

$\boxed{\vec{a}=[a_1,\, a_2,\, a_3]=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}}$

using three *unit vectors* $\hat{i}, \hat{j}, \hat{k}$ along *x*, *y*, *z* axes

$\hat{i}=[1,0,0]$, $\hat{j}=[0,1,0]$, $\hat{k}=[0,0,1]$

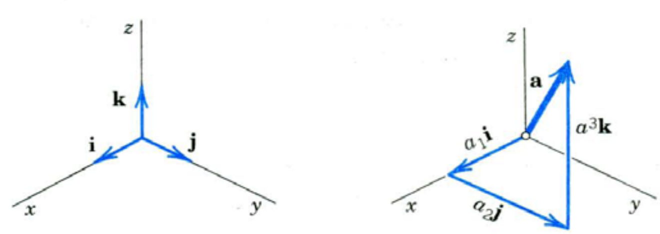


Fig. 175. The unit vectors **i, j, k** and the representation (8)

Example 2 Vector Addition. Multiplication by Scalars

Let

$\vec{a} = [4, 0, 1]$ and $\vec{b} = [2, -5, \frac{1}{3}]$

Then

$-\vec{a} = [-4, 0, -1]$,

$7\vec{a} = [28, 0, 7]$

$\vec{a} + \vec{b} = [6, -5, \frac{4}{3}]$

$2(\vec{a} - \vec{b}) = 2[2, 5, \frac{2}{3}] = 2\vec{a} - 2\vec{b}$

Example 3 $\hat{i}, \hat{j}, \hat{k}$ Notation for Vectors

The two vectors in Example 2 are

$\vec{a} = 4\hat{i} + \hat{k}$

$\vec{b} = 2\hat{i} - 5\hat{j} + \frac{1}{3}\hat{k}$

9.2 Inner Product (Dot Product)

Definition

Inner Product (*Dot Product*) of Vectors is defined as

$$\boxed{\vec{a} \bullet \vec{b} = |\vec{a}||\vec{b}|\cos\gamma = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

$0 \le \gamma \le \pi$, the angle between $\vec{a}$ and $\vec{b}$

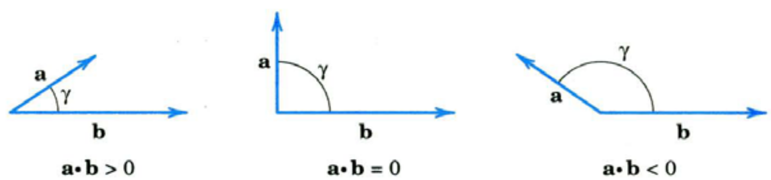


Fig. 176. Angle between vectors and value of inner product

Orthogonality

$\vec{a}$ is orthogonal to $\vec{b}$ if $\vec{a} \bullet \vec{b} = 0$.

γ should be $\pi/2$ when $\vec{a} \ne 0$ and $\vec{b} \ne 0$

Theorem 1 Orthogonality

$\vec{a} \bullet \vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other

Length and Angle

$$\boxed{|\vec{a}| = \sqrt{\vec{a} \bullet \vec{a}}}$$

Then

$$\boxed{\cos\gamma = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a} \bullet \vec{b}}{\sqrt{\vec{a} \bullet \vec{a}}\sqrt{\vec{b} \bullet \vec{b}}}}$$

Basic Properties of Inner Product

$\left(q_1\vec{a}+q_2\vec{b}\right)\bullet\vec{c}=q_1\vec{a}\bullet\vec{c}+q_2\vec{b}\bullet\vec{c}$: linearity

$\vec{a}\bullet\vec{b}=\vec{b}\bullet\vec{a}$: symmetry

$\vec{a}\bullet\vec{a}\geq 0$: positive definiteness

$\vec{a}\bullet\vec{a}=0$ if and only if $\vec{a}=0$: positive definiteness

$\left(\vec{a}+\vec{b}\right)\bullet\vec{c}=\vec{a}\bullet\vec{c}+\vec{b}\bullet\vec{c}$: distributive

$\left|\vec{a}\bullet\vec{b}\right|\leq\left|\vec{a}\right|\left|\vec{b}\right|$: Cauchy-Schwarz inequality

$\left|\vec{a}+\vec{b}\right|\leq\left|\vec{a}\right|+\left|\vec{b}\right|$: triangle inequality

$\left|\vec{a}+\vec{b}\right|^2+\left|\vec{a}-\vec{b}\right|^2=2\left(\left|\vec{a}\right|^2+\left|\vec{b}\right|^2\right)$: parallelogram equality

Example 1 Inner Product. Angle between Vectors

Two vectors $\vec{a}=[1,\ 2,\ 0]$ and $\vec{b}=[3,\ -2,\ 1]$ are given

$$\vec{a}\bullet\vec{b}=1\cdot3+2\cdot(-2)+0\cdot1\Rightarrow -1$$
$$\left|\vec{a}\right|=\sqrt{\vec{a}\bullet\vec{a}}\Rightarrow\sqrt{1^2+2^2+0^2}\Rightarrow\sqrt{5}$$
$$\left|\vec{b}\right|=\sqrt{\vec{b}\bullet\vec{b}}\Rightarrow\sqrt{3^2+(-2)^2+1^2}\Rightarrow\sqrt{14}$$
$$\gamma=\arccos\frac{\vec{a}\bullet\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}=\cos^{-1}\frac{-1}{\sqrt{5}\sqrt{14}}=1.69061=96.865^o$$

Example 2 Work Done by a Force

A constant force $\vec{p}$ is exerted on a body.

But the body is displaced along a vector $\vec{d}$.

Then the work done by the force in the displacement of the body is

$$W=\left(\left|\vec{p}\right|\cos\alpha\right)\left|\vec{d}\right|\Rightarrow\underset{\uparrow}{\vec{p}\bullet\vec{d}}$$

Inner product is used nicely here.

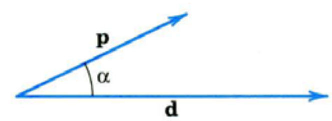


Fig. 177. Work done by a force

Example 3 Component of a Force in a Given Direction

What force in the rope will hold the car on a 25° ramp.

The weight of the car is 5000 lb.

Since the weight points downward,

it can be represented by a vector as

$$\vec{a}=[0,\ -5000,\ 0]$$

$\vec{a}$ can be given by a sum of two vectors

$$\vec{a}=\vec{c}+\vec{p}$$

↑ ↑ Force exerted to the rope by the car

Force exerted to the ramp by the car

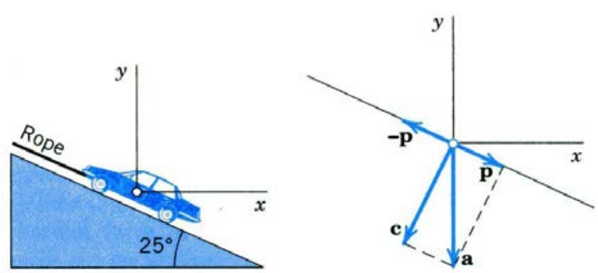


Fig. 178. Example 3

From the figure, $\left|\vec{p}\right|=\left|\vec{a}\right|\cos\left(65^o\right)\Rightarrow 2113\ lb$

- A vector in the direction of the rope $\vec{b}=\left[-1,\ \tan25^o,\ 0\right]$

The force on the rope $\left|\vec{p}\right|=\vec{a}\bullet\left(-\frac{\vec{b}}{\left|\vec{b}\right|}\right)\Rightarrow 2113\ lb$

- *Projection* (or *component*) of $\vec{a}$ in the direction of $\vec{b}$

$$\boxed{p = |\vec{a}|\cos\gamma = \frac{\vec{a}\bullet\vec{b}}{|\vec{b}|}}$$

p is the <u>length</u> of the <u>orthogonal projection</u> of $\vec{a}$ onto $\vec{b}$.

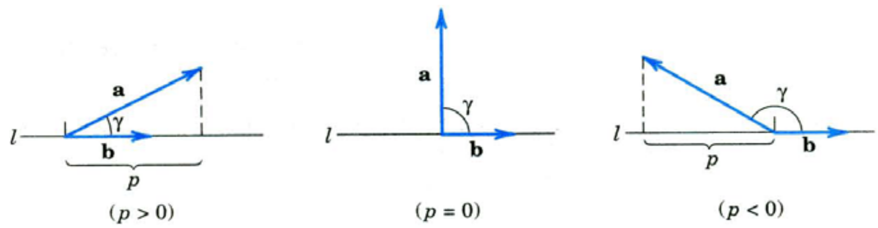
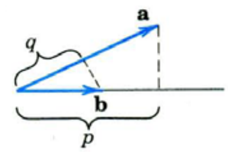


Fig. 179. Component of a vector **a** in the direction of a vector **b**



Projections p of **a** on **b** and q of **b** on **a**

- Orthonormal Basis
  The <u>orthogonal unit vectors</u> in Cartesian coordinates system form an *orthonormal basis* for 3-space.
  $\uparrow$
  $(\hat{i}, \hat{j}, \hat{k})$

  An arbitrary vector is given by a linear combination of the orthonormal basis.

  The coefficients of a vector can be determined by the orthonormality.

$$\vec{v} = l_1\hat{i} + l_2\hat{j} + l_3\hat{k} \quad \rightarrow \quad l_1 = \vec{v}\bullet\hat{i}, \quad l_2 = \vec{v}\bullet\hat{j}, \quad l_3 = \vec{v}\bullet\hat{k}$$

$$\uparrow$$

$$= l_1\hat{i}\bullet\hat{i} + l_2\hat{j}\bullet\hat{i} + l_3\hat{k}\bullet\hat{i}$$

<u>Example 5</u> Orthogonal Straight Lines in the Plane

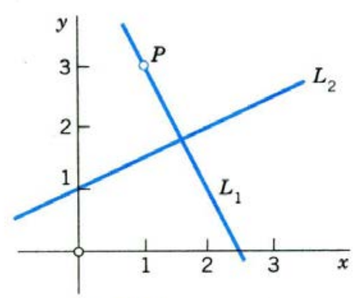


Fig. 181. Example 5

Find the straight line L_1 passing through the point P: (1, 3) in the xy-plane and perpendicular to the straight line L_2: x-2y+2=0

The equation of the straight line L_2 : $b_1x + b_2y = k$
$\uparrow$
$\vec{b}\bullet\vec{r} = k$, in vector form.
$\vec{b} = [b_1, b_2, 0]$, $\vec{r} = [x, y, 0]$

Consider another straight line $\vec{b}\bullet\vec{r} = 0$
$\rightarrow$ This line passes through the origin and parallel to L_2
$\rightarrow$ $\vec{r}$ is the position vector from the origin to a point on L_2.
Since $\vec{b}\bullet\vec{r} = 0$, $\vec{b}$ is normal to $\vec{r}$ and to this line and to L_2.

The equation of a line parallel to $\vec{b}$ is $\vec{a}\bullet\vec{r} = c$ with $\vec{a}\bullet\vec{b} = 0$.
Since $\vec{b} = [1, -2, 0]$ from L_2, $\vec{a} = [2, 1, 0]$
$\rightarrow$ L_1 : $2x + y = c$

L_1 passes through P: (1, 3)
$\rightarrow$ 2+3=c $\rightarrow$ L_1 : $2x + y = 5$

Example 6 Normal Vector to a Plane

Find a unit vector normal to the plane $4x+2y+4z=-7$

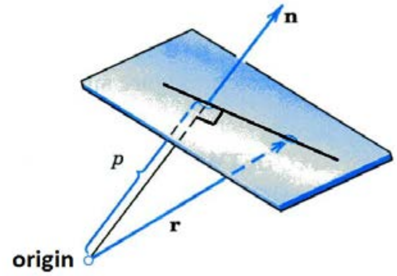


Fig. 182. Normal vector to a plane

Express the plane in vector form : $\vec{a}\bullet\vec{r}=c$

Using the unit vector of $\vec{a}$, $\hat{n}=\dfrac{\vec{a}}{|\vec{a}|}$

$\underset{\substack{\uparrow\\ \text{Projection of } \vec{r} \text{ onto } \hat{n}}}{\hat{n}\bullet\vec{r}=p}$: $p=c/|\vec{a}|$, a constant.

A position vector $\vec{r}$ is from the origin to a point on the plane.

The same projection p for any $\vec{r}$ $\rightarrow$ $\hat{n}$ should be a surface normal.

Since $\vec{a}=[4,\ 2,\ 4]$ is given, the surface normal is obtained as $\hat{n}=\dfrac{\vec{a}}{|\vec{a}|}\Rightarrow\dfrac{1}{6}\vec{a}$

9.3 Vector Product (Cross Product, Outer Product)

Definition

The *vector product* of $\vec{a}$ and $\vec{b}$ is defined as

$\vec{v}=\vec{a}\times\vec{b}$: another vector

Its magnitude

$|\vec{v}|=|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\gamma$: γ, angle between $\vec{a}$ and $\vec{b}$

Its direction is perpendicular to both $\vec{a}$ and $\vec{b}$ conforming to *right-handed triple*(or *screw*).

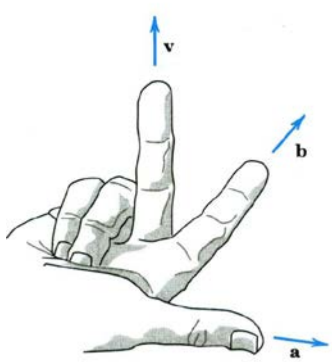


Fig. 184. Right-handed triple of vectors **a, b, v**

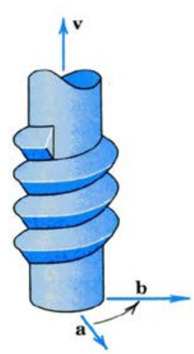


Fig. 185. Right-handed screw

Note that $|\vec{a}\times\vec{b}|$ represents the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$.

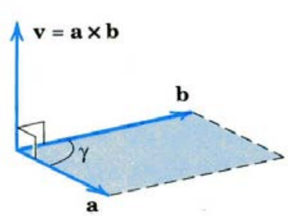


Fig. 183. Vector product

- In components

$$\boxed{\vec{a}\times\vec{b}=\left[a_2b_3-a_3b_2\,,\ a_3b_1-a_1b_3\,,\ a_1b_2-a_2b_1\right]}$$

$\vec{v}=\vec{a}\times\vec{b}$ can be calculated as follows

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\Rightarrow \hat{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-\hat{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+\hat{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

Example 1 Vector Product

$\vec{a}=[1,\ 1,\ 0]$, $\vec{b}=[3,\ 0,\ 0]$

The vector product

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 0\\ 3 & 0 & 0\end{vmatrix}=\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & 0\\ 3 & 0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & 1\\ 3 & 0\end{vmatrix}\mathbf{k}=-3\mathbf{k}=[0,\ 0,\ -3].$$

Example 2 Vector Products of the Standard Basis Vectors

$$\mathbf{i}\times\mathbf{j}=\mathbf{k},\qquad \mathbf{j}\times\mathbf{k}=\mathbf{i},\qquad \mathbf{k}\times\mathbf{i}=\mathbf{j}$$

$$\mathbf{j}\times\mathbf{i}=-\mathbf{k},\qquad \mathbf{k}\times\mathbf{j}=-\mathbf{i},\qquad \mathbf{i}\times\mathbf{k}=-\mathbf{j}.$$

Theorem 1 General Properties of Vector Products

$(k\vec{a})\times\vec{b}=k(\vec{a}\times\vec{b})=\vec{a}\times k\vec{b}$	: for every scalar *k*
$\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$	: distributive
$(\vec{a}+\vec{b})\times\vec{c}=(\vec{a}\times\vec{c})+(\vec{b}\times\vec{c})$	: distributive
$\vec{b}\times\vec{a}=-(\vec{a}\times\vec{b})$	: anticommutative
$\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$	: not associative

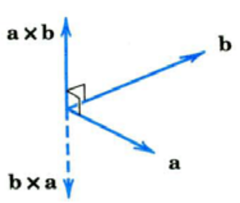


Fig. 187.
Anticommutativity

Example 3   Moment of Force

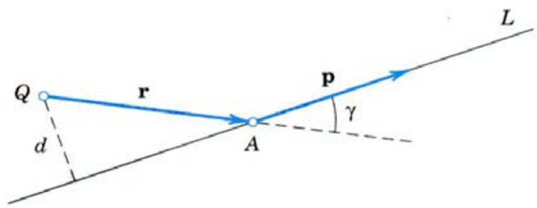


Fig. 188. Moment of a force **p**

A force $\vec{p}$ is exerted on a point A.
Point *Q* and point *A* is connected by a vector $\vec{r}$.

The *moment* *m* about a point *Q* is defined as $m = |\vec{p}|d$,
where *d* is the perpendicular distance from *Q* to *L*.
$\rightarrow \quad m = |\vec{p}||\vec{r}|\sin\gamma$

In vector form
$\vec{m} = \vec{r} \times \vec{p}$: *Moment vector*

Example 5   Velocity of a Rotating Body

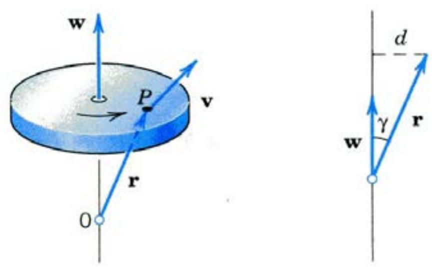


Fig. 190. Rotation of a rigid body

A vector $\vec{w}$ can describe a rotation of a rigid body.
$\rightarrow$ Its direction $\parallel$ the rotation axis (right-hand rule)
Its magnitude = angular speed ω (radian/sec)

The linear speed at a point *P*
$\rightarrow \quad v = \omega d \Rightarrow |\vec{w}||\vec{r}|\sin\gamma \Rightarrow |\vec{w} \times \vec{r}|$

In vector form
$\vec{v} = \vec{w} \times \vec{r}$

Scalar Triple Product

The *scalar triple product* is defined as

$$\boxed{\left(\vec{a}\ \vec{b}\ \vec{c}\right)=\vec{a}\bullet\left(\vec{b}\times\vec{c}\right)}$$

It can be calculated as

$$\boxed{\vec{a}\bullet\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}}$$

Theorem 2 Properties and Applications of Scalar Triple Products

(a) The dot and cross can be interchanged

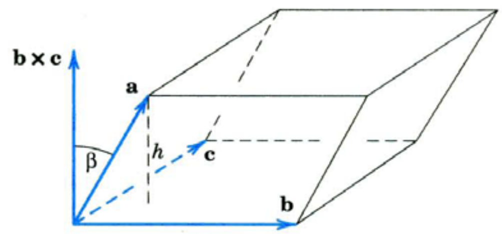
$$\vec{a}\bullet\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\bullet\vec{c}$$

(b) Its absolute value is the volume of the parallelepiped formed by $\vec{a}$, $\vec{b}$ and $\vec{c}$.

(c) Any three vectors are linearly independent
if and only if their scalar tipple product is nonzero.

Proof:

(a) It can be proved by direct calculations



(b) $\left|\vec{a}\bullet\left(\vec{b}\times\vec{c}\right)\right|=|\vec{a}|\left|\vec{b}\times\vec{c}\right||\cos\beta|\Rightarrow\left(|\vec{a}||\cos\beta|\right)\left|\vec{b}\times\vec{c}\right|$

↑ height of the parallelepiped ↑ area of the base

(c) If three vectors are in the same plane or on the same straight line,
either the dot or cross product becomes zero in $\vec{a}\bullet\left(\vec{b}\times\vec{c}\right)$.

$\vec{a}\bullet\left(\vec{b}\times\vec{c}\right)\neq 0 \quad\rightarrow\quad$ Three vectors NOT in the same plane or on the same straight line.
They are linearly independent.

Example 6 Tetrahedron

A tetrahedron is formed by three edge vectors,

$$\vec{a}=[2,\ 0,\ 3],\quad \vec{b}=[0,\ 4,\ 1],\quad \vec{c}=[5,\ 6,\ 0]$$

Find its volume.

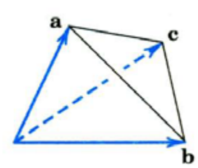


Fig. 192. Tetrahedron

First, find the volume of the parallelepiped using scalar triple product.

$$(\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c})=\begin{vmatrix}2 & 0 & 3\\ 0 & 4 & 1\\ 5 & 6 & 0\end{vmatrix}=2\begin{vmatrix}4 & 1\\ 6 & 0\end{vmatrix}+3\begin{vmatrix}0 & 4\\ 5 & 6\end{vmatrix}=-12-60=-72.$$

The volume of tetrahedron is 1/6 of that of parallelepiped.
Therefore, the answer is 12.

9.4 Vector and Scalar Functions and Fields.

A *vector function* gives a vector value for a point p in space

$$\vec{v} = \vec{v}(p) = [v_1(p),\ v_2(p),\ v_3(p)] \xrightarrow{\text{In Cartesian Coord.}} \vec{v}(x,\ y,\ z) = [v_1(x,\ y,\ z),\ v_2(x,\ y,\ z),\ v_3(x,\ y,\ z)]$$

A *scalar function* gives scalar values : $f = f(p)$

A vector function defines a *vector field*.
A scalar function defines a *scalar field*.

In Engineering

Meaning of field = Meaning of function.

The field implies spatial distribution of a quantity.

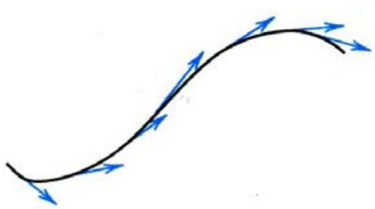


Fig. 193. Field of tangent vectors of a curve

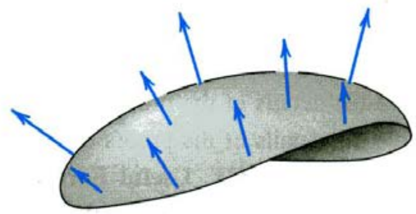


Fig. 194. Field of normal vectors of a surface

Example 1 Scalar function

The distance from a fixed point p_o to any point p is a scalar function, $f(p)$.

$f(p)$ defines a scalar field in space $\rightarrow$ It means that the scalar values are distributed in space.

$$f(p) = f(x,y,x) = \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2}$$

In a different coordinate system

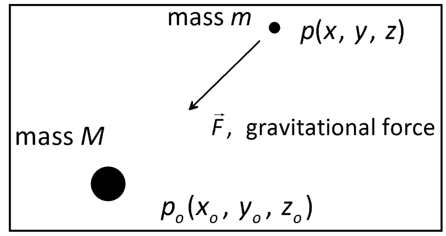
$\rightarrow$ p and p_o have different forms, but $f(p)$ has the same value.

$\rightarrow$ $f(p)$ is a scalar function.

Direction cosines of the line from p_o to p depend on the choice of coordinate system.

$\rightarrow$ Not a scalar function.

Example 3 Vector Field (Gravitation field)



Newton's law of gravitation

$$|\vec{F}| = \frac{c}{r^2} \qquad : \ r = \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2}$$

The direction of $\vec{F}$ is from p to p_o.

$\vec{F}(x,y,z)$ defines a vector field in space.

- In vector form
  Define the position vector,

$$\vec{r} \equiv (x-x_o)\hat{i} + (y-y_o)\hat{j} + (z-z_o)\hat{k}$$: Its direction is from p_o to p.

Then

$$\vec{F} = -\frac{c}{r^3}\vec{r}$$

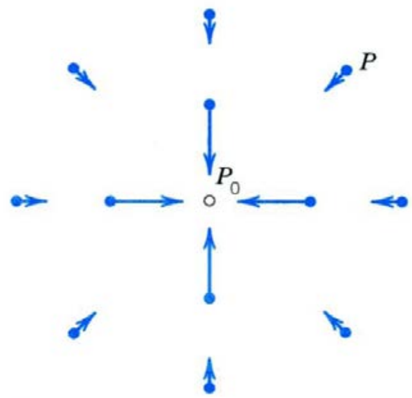


Fig. 196. Gravitational field in Example 3

Vector Calculus

Convergence

An infinite sequence of vectors $\vec{a}_{(1)}, \vec{a}_{(2)}, \vec{a}_{(3)}, \ldots$ *converges* to $\vec{a}$ if

$$\lim_{n\to\infty} \left|\vec{a}_{(n)} - \vec{a}\right| = 0 .$$

$$\rightarrow \quad \lim_{n\to\infty} \vec{a}_{(n)} = \vec{a} \qquad : \ \vec{a}\text{, } \textit{limit vector}$$

Similarly, a vector function $\vec{v}(t)$ has the limit $\vec{l}$ at t_o if

$$\lim_{t\to t_o} \left|\vec{v}(t) - \vec{l}\right| = 0 .$$

$$\rightarrow \quad \lim_{t\to t_o} \vec{v}(t) = \vec{l}$$

Continuity

$\vec{v}(t)$ is continuous at $t = t_o$ if

$$\lim_{t\to t_o} \vec{v}(t) = \vec{v}(t_o)$$

$\vec{v}(t)$ is continuous at $t = t_o$ if and only if its three components are continuous at t_o

$$\vec{v}(t) = v_1(t)\hat{i} + v_2(t)\hat{j} + v_3(t)\hat{k}$$

Definition Derivatives of a Vector Function

$\vec{v}(t)$ is *differentiable* at *t* if the limit exists

$$\vec{v}\,'(t)=\lim_{\Delta t\to 0}\frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t}$$

$\uparrow$

Called *derivative* of $\vec{v}(t)$

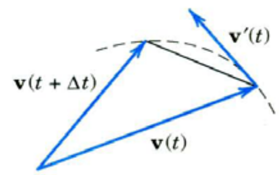


Fig. 197. Derivative of a vector function

$\vec{v}(t)$ is differentiable at *t* if and only if its three components are differentiable at *t*.

$\rightarrow \quad \vec{v}\,'(t)=\left[v_1'(t),\ v_2'(t),\ v_3'(t)\right]$

- Differentiation rules

$(c\vec{v})'=c\vec{v}\,'$: *c*, constant

$(\vec{u}+\vec{v})'=\vec{u}'+\vec{v}'$

$(\vec{u}\bullet\vec{v})'=\vec{u}'\bullet\vec{v}+\vec{u}\bullet\vec{v}'$

$(\vec{u}\times\vec{v})'=\vec{u}'\times\vec{v}+\vec{u}\times\vec{v}'$

$(\vec{u}\ \vec{v}\ \vec{w})'=(\vec{u}'\ \vec{v}\ \vec{w})+(\vec{u}\ \vec{v}'\ \vec{w})+(\vec{u}\ \vec{v}\ \vec{w}')$

Example 4 Derivative of a Vector Function of Constant Length

Let $\vec{v}(t)$ be a vector function with a constant length, $|\vec{v}(t)|=c$.

$$|\vec{v}(t)|^2=\vec{v}(t)\bullet\vec{v}(t)=c^2$$

The total derivative

$$(\vec{v}\bullet\vec{v})'=2\vec{v}\bullet\vec{v}\,'=0$$

$\uparrow$

$\vec{v}\,'=0$ or $\vec{v}\perp\vec{v}\,'$

Partial Derivatives of a Vector Function

Let the components with two or more variables be differentiable (ex.wind direction w.r.t. time and altitude)

$$\vec{v}=v_1(t_1,\,t_2,\,\ldots\,t_n)\hat{i}+v_2(t_1,\,t_2,\,\ldots\,t_n)\hat{j}+v_3(t_1,\,t_2,\,\ldots\,t_n)\hat{k}$$

The *partial derivative* of $\vec{v}$ with respect to t_m

$$\frac{\partial\vec{v}}{\partial t_m}=\frac{\partial v_1}{\partial t_m}\hat{i}+\frac{\partial v_2}{\partial t_m}\hat{j}+\frac{\partial v_3}{\partial t_m}\hat{k}$$

The second partial derivative

$$\frac{\partial^2\vec{v}}{\partial t_l\partial t_m}=\frac{\partial^2 v_1}{\partial t_l\partial t_m}\hat{i}+\frac{\partial^2 v_2}{\partial t_l\partial t_m}\hat{j}+\frac{\partial^2 v_3}{\partial t_l\partial t_m}\hat{k}$$

Example 5 Partial derivatives

$$\mathbf{r}(t_1,\,t_2)=a\cos t_1\,\mathbf{i}+a\sin t_1\,\mathbf{j}+t_2\mathbf{k}.$$

$$\frac{\partial\mathbf{r}}{\partial t_1}=-a\sin t_1\,\mathbf{i}+a\cos t_1\,\mathbf{j}\quad\text{and}\quad\frac{\partial\mathbf{r}}{\partial t_2}=\mathbf{k}.$$