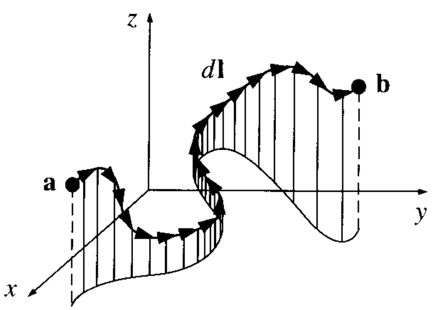
Chapter 1. Vector Analysis

- Vector Algebra
- Differential Calculus
- Integral Calculus
- Curvilinear Coordinates
- The Dirac Delta Function
- The Theory of Vector Fields

1.3 Integral Calculus (적분학)

1.3.1 Line, Surface, and Volume Integrals (선적분, 면적분, 체적분)

- **선적분 (Line Integrals)**:



$$\int_a^b \vec{E}\cdot d\vec{l}$$

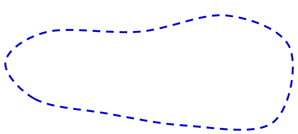
where $\vec{E}$ is a vector,

$d\vec{l}$ is the infinitesimal displacement vector (극소 변위 벡터)

$$= \int_a^b E\,dl\cos\theta$$

$$= \int_a^b \left(E_x\hat{x} + E_y\hat{y} + E_z\hat{z}\right)\cdot\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right) = \int_a^b \left(E_x dx + E_y dy + E_z dz\right)$$

- If the path forms a closed loop (*i.e.*, if $b = a$) [만약 경로가 닫힌 경로이라면],



$$\oint \vec{E}\cdot d\vec{l}$$

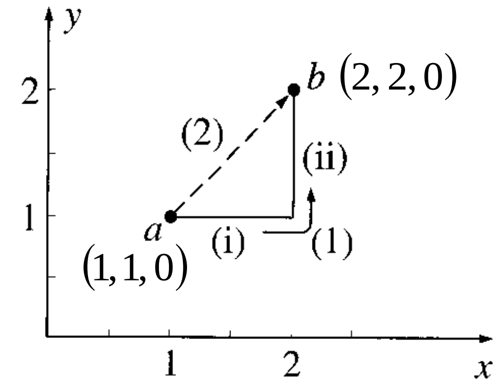
[Example] Work done by a force $\vec{\mathrm{F}}$ (힘 $\vec{\mathrm{F}}$ 에 의한 일): $W = \int \vec{F}\cdot d\vec{l}$

1.3 Integral Calculus (적분학)

[Example 1.6] $\vec{E} = y^2\hat{x} + 2x(y+1)\hat{y}$

Calculate the line integral of **v** along the paths (1) and (2)

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$



경로 (1) : (i) $dx: x = 1 \rightarrow 2$ at $y = 1$

(ii) $dy: y = 1 \rightarrow 2$ at $x = 2$

$$\int_a^b \vec{E}\cdot d\vec{l} = \int_a^b \left[y^2\hat{x} + 2x(y+1)\hat{y}\right]\cdot(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z})$$

$$= \int_a^b \left[y^2 dx + 2x(y+1)dy\right] = \left[\int_{x=1}^{2} y^2 dx\right]_{y=1} + \left[\int_{y=1}^{2} 2x(y+1)dy\right]_{x=2}$$

$$= \left[y^2\{x\}_{x=1}^{x=2}\right]_{y=1} + \left[2x\left(\frac{y^2}{2} + y\right)_{y=1}^{y=2}\right]_{x=2}$$

$$= \left[y^2(2-1)\right]_{y=1} + \left[2x\left(\frac{4}{2} + 2 - \frac{1}{2} - 1\right)\right]_{x=2}$$

$$= [1\cdot 1] + \left[4\cdot\left(\frac{3}{2} + 1\right)\right] = 1 + 10 = 11$$

경로 (2) : $(1,1) \rightarrow (2,2)$

$y = x \longrightarrow dy = dx$

$$\int_a^b \vec{E}\cdot d\vec{l} = \int_{a=(1,1)}^{b=(2,2)} \left[y^2 dx + 2x(y+1)dy\right]$$

$$= \int_{x=1}^{2} \left[x^2 dx + 2x(x+1)dx\right]$$

$$= \int_{x=1}^{2} (3x^2 + 2x)dx$$

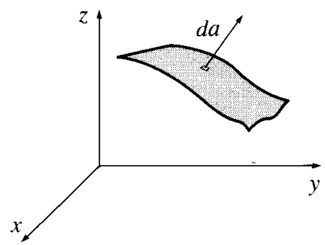
$$= \left[x^3\right]_{x=1}^{2} + \left[x^2\right]_{x=1}^{2}$$

$$= [8-1] + [4-1] = 7 + 3 = 10$$

경로 (1)과 경로(2)를 통한 한 바퀴 순환의 경우 $\oint \vec{E}\cdot d\vec{l} = 11 - 10 = 1$

1.3 Integral Calculus (적분학)

- 면적분 (Surface Integrals):



$$\int_S \vec{E} \cdot d\vec{a}$$

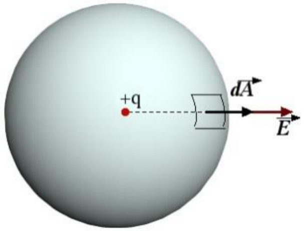
where $\vec{E}$ is a vector,

$d\vec{a}$ is the infinitesimal patch of area (극소 면적 벡터)

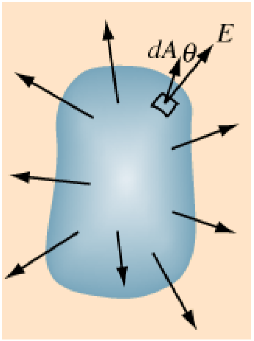
$$= \int_a^b E\, da \cos\theta$$

$$= \int_a^b \left(E_x \hat{x} + E_y \hat{y} + E_z \hat{z}\right) \cdot \left(da_x\, \hat{x} + da_y\, \hat{y} + da_z\, \hat{z}\right) = \int_a^b \left(E_x da_x + E_y da_y + E_z da_z\right)$$

- If the surface is closed (forming a “balloon”) [만약 풍선처럼 닫힌 면이라면],



$$\oint \vec{E} \cdot d\vec{a}$$

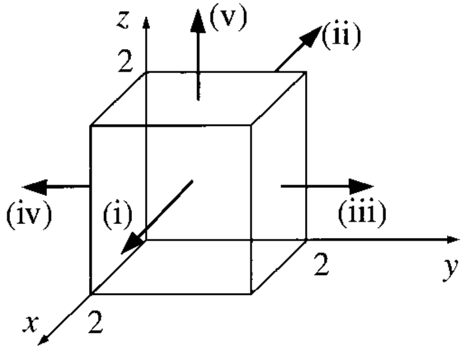


적분값은 통상 바깥쪽으로 나가는(outward) 방향일 때 “양(positive)” 값이며,

단위 시간당 바깥쪽으로 나가는 **유량** 또는 **유속** (flux)을 의미함.

1.3 Integral Calculus (적분학)

[Example 1.7] $\vec{E} = 2xz\,\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$



Calculate the surface integral of $\vec{E}$ over five sides (excluding the bottom) of the cubical box.
(Let "upward and outward" be the positive directions.)

For the surface (i) $x=2,\ d\vec{a} = dydz\,\hat{x}$

$$\vec{E}\cdot d\vec{a} = \left[2xz\,\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}\right]\cdot\left[dydz\,\hat{x}\right]_{x=2} = \left[2xz\,dy\,dz\right]_{x=2} = 4z\,dy\,dz$$

$$\int_{S(\mathrm{i})}\vec{E}\cdot d\vec{a} = \int_{z=0}^{2}\int_{y=0}^{2} 4z\,dydz = \int_{y=0}^{2} dy\int_{z=0}^{2} 4z\,dz = \left[y\right]_{y=0}^{2}\left[2z^2\right]_{z=0}^{2} = 2\times 8 = 16$$

For the surface (ii) $x=0,\ d\vec{a} = -dydz\,\hat{x},\quad \vec{E}\cdot d\vec{a} = \left[-2xz\,dy\,dz\right]_{x=0} = 0 \longrightarrow \int_{S(\mathrm{ii})}\vec{E}\cdot d\vec{a} = 0$

For the surface (iii) $y=2,\ d\vec{a} = dxdz\,\hat{y},$

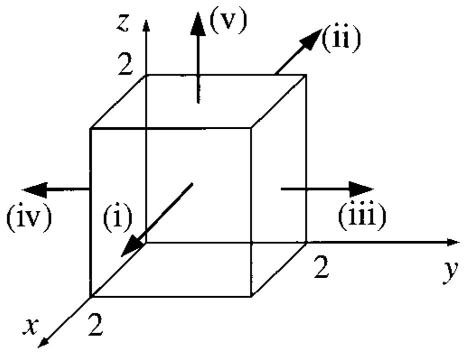
$$\vec{E}\cdot d\vec{a} = \left[2xz\,\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}\right]\cdot\left[dxdz\,\hat{y}\right]_{y=2} = \left[(x+2)\,dx\,dz\right]_{y=2} = (x+2)\,dx\,dz$$

$$\int_{S(\mathrm{iii})}\vec{E}\cdot d\vec{a} = \int_{z=0}^{2}\int_{x=0}^{2}(x+2)\,dxdz = \int_{x=0}^{2}(x+2)dx\int_{z=0}^{2}dz = \left[\frac{x^2}{2}+2x\right]_{x=0}^{2}\left[z\right]_{z=0}^{2} = 6\times 2 = 12$$

1.3 Integral Calculus (적분학)

[Example 1.7] *(continued)*

$$\vec{E} = 2xz\,\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$$



For the surface (iv) $y = 0,\ d\vec{a} = -dxdz\,\hat{y},$

$$\vec{E}\cdot d\vec{a} = \left[-(x+2)\,dx\,dz\right]_{y=0} = -(x+2)\,dx\,dz$$

$$\int_{S(\text{iv})} \vec{E}\cdot d\vec{a} = \int_{z=0}^{2}\int_{x=0}^{2} -(x+2)\,dxdz = -\int_{x=0}^{2}(x+2)dx\int_{z=0}^{2}dz = -\left[\frac{x^2}{2}+2x\right]_{x=0}^{2}[z]_{z=0}^{2}$$

$$= -6\times 2 = -12$$

For the surface (v) $z = 2,\ d\vec{a} = dxdy\,\hat{z},$

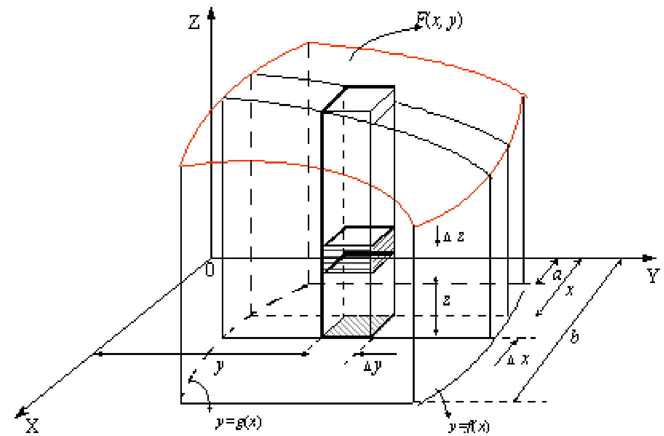
$$\vec{E}\cdot d\vec{a} = \left[2xz\,\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}\right]\cdot\left[dxdy\,\hat{z}\right]_{x=2} = \left[y(z^2-3)\,dx\,dy\right]_{z=2} = y\,dx\,dy$$

$$\int_{S(\text{v})} \vec{E}\cdot d\vec{a} = \int_{x=0}^{2}\int_{y=0}^{2} y\,dxdy = -\int_{x=0}^{2}dx\int_{z=0}^{2}y\,dy = [x]_{x=0}^{2}\left[\frac{y^2}{2}\right]_{z=0}^{2} = 2\times 2 = 4$$

Total flux (총 유속): $\int_S \vec{E}\cdot d\vec{a} = 16 + 0 + 12 - 12 + 4 = 20.$

1.3 Integral Calculus (적분학)

- **체적분 (Volume Integrals)**:

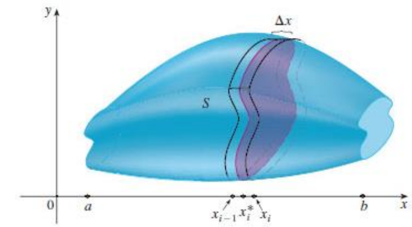


$$\int_V T\,dV$$

where T is a scalar function,

dV is the infinitesimal volume element (극소 부피 성분)

$$dV = dx\,dy\,dz \qquad \text{(in Cartesian coordinates)}$$



- If T is an non-uniform density of a substance (T가 물질의 불균일한 밀도이라면) ,

$$\int_V T\,dV = M \qquad \text{: Total mass (총 질량).}$$

- If a vector **v** is used instead of T (스칼라 량인 T 대신에 벡터 **v**가 사용되면) ,

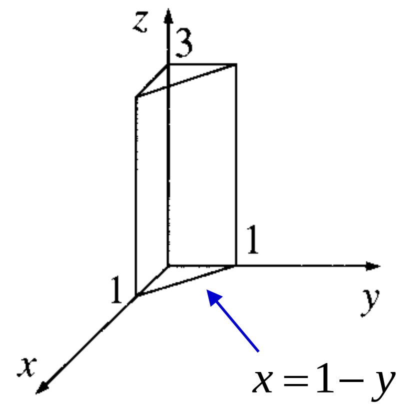
$$\int_V \vec{v}\,dV = \int_V \left(v_x\,\hat{x} + v_y\,\hat{y} + v_z\,\hat{z}\right) dV = \hat{x}\int v_x\,dV + \hat{y}\int v_x\,dV + \hat{z}\int v_x\,dV$$

1.3 Integral Calculus (적분학)

[Example 1.8]

$$T = xyz^2$$

Calculate the volume integral of T over the prism.



$$\int_V T\,dV = \int_V T\,dx\,dy\,dz = \int_{z=0}^{3}\int_{y=0}^{1}\int_{x=0}^{1-y} xyz^2\,dx\,dy\,dz$$

$$= \int_{z=0}^{3} z^2 \left\{ \int_{y=0}^{1} y \left[\int_{x=0}^{1-y} x\,dx \right] dy \right\} dz$$

$$= \int_{z=0}^{3} z^2 \left\{ \int_{y=0}^{1} y \frac{(1-y)^2}{2} dy \right\} dz$$

$$= \frac{1}{2}\int_{z=0}^{3} z^2 dz \int_{y=0}^{1} \left(y - 2y^2 + y^3\right) dy$$

$$= \frac{1}{2}\left[\frac{z^3}{3}\right]_{z=0}^{3} \left[\frac{y^2}{2} - \frac{2}{3}y^3 + \frac{y^4}{4}\right]_{y=0}^{1} = \frac{1}{2}\cdot 9 \cdot \frac{1}{12} = \frac{3}{4}$$

1.3 Integral Calculus (적분학)

1.3.2 The Fundamental Theorem of Calculus (적분의 기본 정리)

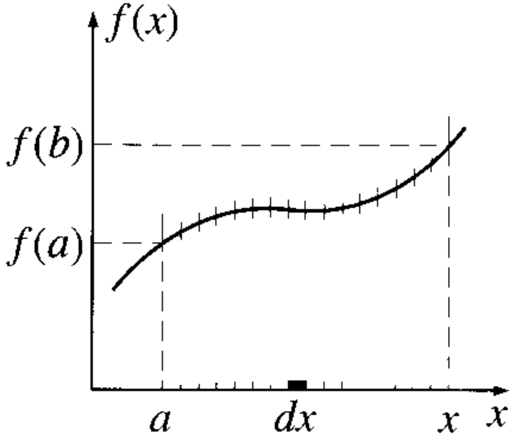
1개의 변수를 가진 함수 $f(x)$ 의 경우

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$$

or

$$\int_a^b F(x) dx = f(b) - f(a)$$

where $$\frac{df}{dx} = F(x)$$



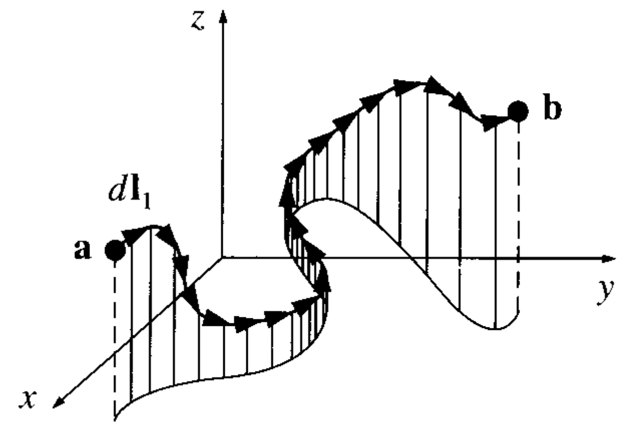
1.3 Integral Calculus (적분학)

1.3.3 The Fundamental Theorem of Gradients (Gradient의 기본 정리)

3개의 변수를 가진 스칼라 함수 $T(x,y,z)$의 경우

$$dT = (\nabla T)\cdot d\vec{l}_1$$

The total change in T in going from a to b



$$\int_a^b (\nabla T)\cdot d\vec{l} = T(b) - T(a)$$

: The fundamental theorem of gradients

allows the line integrals to be path independent.

$$= \int_a^b \nabla T \, dl \cos\theta$$

$$= \int_a^b \left(\hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z} \right)\cdot\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right) = \int_a^b \left(\frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz \right)$$

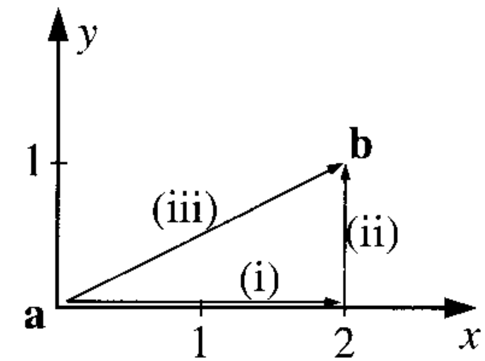
[Corollary 1 (따름 정리1)] $\int_a^b (\nabla T)\cdot d\vec{l}$ is **independent** on the path taken from ***a*** to ***b***.

[Corollary 2 (따름 정리2)] $\oint (\nabla T)\cdot d\vec{l} = 0$ since ***a*** is identical to ***b***

[i.e., T(b) – T(a) = 0].

1.3 Integral Calculus (적분학)

[Example 1.9] $T = xy^2$



take point ***a*** to be the origin (0, 0, 0) and ***b*** the point (2, 1, 0).

Check the fundamental theorem for gradients.

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$\nabla T = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(xy^2) = y^2\,\hat{x} + 2xy\,\hat{y}$$

For path (i) $y = 0;\ d\vec{l} = dx\,\hat{x},\ \nabla T \cdot d\vec{l} = (y^2\,\hat{x} + 2xy\,\hat{y}) \cdot dx\,\hat{x} = y^2 dx = 0 \rightarrow \int_{\mathrm{i}} \nabla T \cdot d\vec{l} = 0$

For path (ii) $x = 2;\ d\vec{l} = dy\,\hat{y},\ \nabla T \cdot d\vec{l} = (y^2\,\hat{x} + 2xy\,\hat{y}) \cdot dy\,\hat{y} = 2xy\,dy = 4y\,dy$

$$\int_{\mathrm{ii}} \nabla T \cdot d\vec{l} = \int_{y=0}^{1} 4y\,dy = \left[2y^2\right]_{y=0}^{1} = 2$$

For path (i) + path (ii) $\int_a^b (\nabla T) \cdot d\vec{l} = 2 + 0 = 2$

$$T(2,1) - T(0,0) = 2 \cdot 1 - 0 = 2$$

For path (iii) $y = \frac{1}{2}x,\ dy = \frac{1}{2}dx;\ d\vec{l} = dx\,\hat{x} + dy\,\hat{y},$

$$\nabla T \cdot d\vec{l} = (y^2\,\hat{x} + 2xy\,\hat{y}) \cdot (dx\,\hat{x} + dy\,\hat{y}) = y^2 dx + 2xy\,dy = \frac{x^2}{4}dx + \frac{x^2}{2}dx = \frac{3}{4}x^2 dx$$

$$\int_{\mathrm{iii}} \nabla T \cdot d\vec{l} = \int_{x=0}^{2} \frac{3}{4}x^2\,dx = \left[\frac{1}{4}x^3\right]_{x=0}^{2} = 2$$

$$\therefore \int_a^b (\nabla T) \cdot d\vec{l} = T(b) - T(a)$$

1.3 Integral Calculus (적분학)

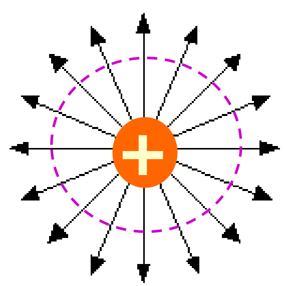
1.3.4 The Fundamental Theorem for Divergences (Divergence의 기본 정리)

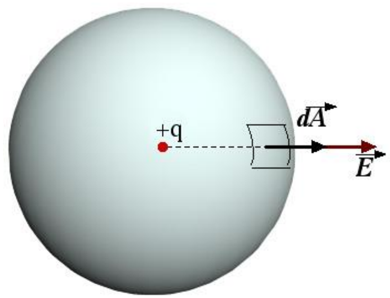
$$\int_V \left(\nabla \cdot \vec{E}\right) dV = \oint_S \vec{E} \cdot d\vec{a}$$

: Gauss's Theorem
: Green's Theorem
: Divergence Theorem

$$\rightarrow \int_V \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) dxdydz = \oint_S \left(E_x da_x + E_y da_y + E_z da_z \right)$$

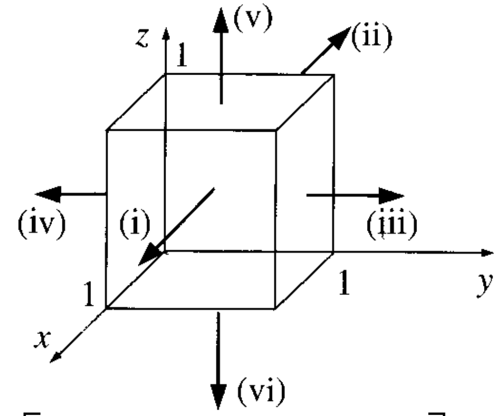
$$\int (\text{주어진공간내의전하 분포들의합}) = \oint (\text{갇힌표면을통해나오는전기장의합})$$





1.3 Integral Calculus (적분학)

[Example 1.10] $\vec{E} = y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$



Check the divergence theorem.

$$\nabla \cdot \vec{E} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left[y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}\right]$$

$$= 2x + 2y = 2(x + y)$$

$$\int_V (\nabla \cdot \vec{E})dV = \int_V 2(x+y)dV = 2\int_{z=0}^{1}\left[\int_{y=0}^{1}\left\{\int_{x=0}^{1}(x+y)dx\right\}dy\right]dz = 2\int_{z=0}^{1}\left[\int_{y=0}^{1}\left\{\frac{x^2}{2} + xy\right\}_{x=0}^{1} dy\right]dz$$

$$= 2\int_{z=0}^{1}\left[\int_{y=0}^{1}\left(\frac{1}{2} + y\right)dy\right]dz = 2\int_{z=0}^{1}\left[\frac{y}{2} + \frac{y^2}{2}\right]_{y=0}^{1} dz = 2\int_{z=0}^{1} 1\,dz = 2 \times 1 = 2$$

$$\oint_S \vec{E} \cdot d\vec{a} =$$

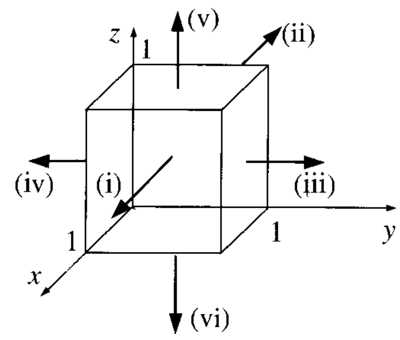
For the surface(i) $x = 1;\ d\vec{a} = dy\,dz\,\hat{x},$

$$\vec{E} \cdot d\vec{a} = \left[y^2\,\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}\right] \cdot dydz\,\hat{x} = y^2 dydz$$

$$\int_{(i)} \vec{E} \cdot d\vec{a} = \int_{z=0}^{1}\int_{y=0}^{1} y^2 dydz = \int_{z=0}^{1}\left[\frac{y^3}{3}\right]_{y=0}^{1} dz = \int_{z=0}^{1}\frac{1}{3}dz = \frac{1}{3}[z]_{z=0}^{1} = \frac{1}{3}$$

1.3 Integral Calculus (적분학)

[Example 1.10] *(continued)* $\vec{E} = y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$



For the surface(ii) $x = 0;\ d\vec{a} = -dy\,dz\,\hat{x},\quad \vec{E}\cdot d\vec{a} = -y^2 dydz$

$$\int_{(\text{ii})}\vec{E}\cdot d\vec{a} = -\int_{z=0}^{1}\int_{y=0}^{1} y^2 dydz = -\frac{1}{3}$$

For the surface(iii) $y = 1;\ d\vec{a} = dx\,dz\,\hat{y},$

$$\vec{E}\cdot d\vec{a} = \left[y^2\ \hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}\right]\cdot dxdz\ \hat{y} = (2xy + z^2)dxdz = (2x + z^2)dxdz$$

$$\int_{(\text{iii})}\vec{E}\cdot d\vec{a} = \int_{z=0}^{1}\int_{x=0}^{1}(2x + z^2)dxdz = \int_{z=0}^{1}\left[x^2 + z^2x\right]_{x=0}^{1} dz = \int_{z=0}^{1}(1 + z^2)dz = \left[z + \frac{z^3}{3}\right]_{z=0}^{1} = \frac{4}{3}$$

For the surface(iv) $y = 0;\ d\vec{a} = -dx\,dz\,\hat{y},\quad \vec{E}\cdot d\vec{a} = -(2xy + z^2)dxdz = -z^2 dxdz$

$$\int_{(\text{iv})}\vec{E}\cdot d\vec{a} = -\int_{z=0}^{1}\int_{x=0}^{1} z^2 dxdz = \int_{z=0}^{1}\left[z^2x\right]_{x=0}^{1} dz = \int_{z=0}^{1} z^2 dz = \left[\frac{z^3}{3}\right]_{z=0}^{1} = \frac{1}{3}$$

For the surface(v) $z = 1;\ d\vec{a} = dx\,dy\,\hat{z},$

$$\vec{E}\cdot d\vec{a} = \left[y^2\ \hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}\right]\cdot dxdy\ \hat{z} = 2yz\ dxdz = 2y\ dxdz$$

$$\int_{(\text{v})}\vec{E}\cdot d\vec{a} = \int_{y=0}^{1}\int_{x=0}^{1} 2y\ dxdy = \int_{z=0}^{1}\left[2xy\right]_{x=0}^{1} dy = \int_{z=0}^{1} 2ydy = \left[y^2\right]_{y=0}^{1} = 1$$

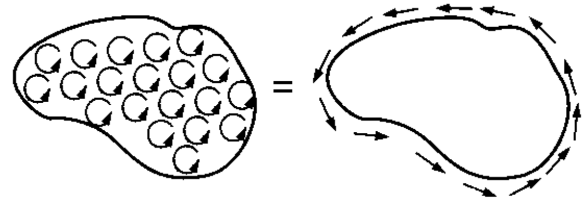
For the surface(vi) $z = 0;\ d\vec{a} = -dx\,dy\,\hat{z},\quad \vec{E}\cdot d\vec{a} = 2yz\ dxdz = 0 \qquad \int_{(\text{v})}\vec{E}\cdot d\vec{a} = 0$

$$\therefore \oint_S \vec{E}\cdot d\vec{a} = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2$$

1.3 Integral Calculus (적분학)

1.3.5 The Fundamental Theorem for Curls (Curl의 기본 정리)

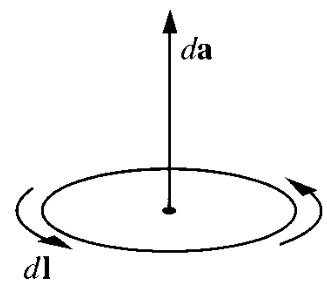
$$\int_S \left(\nabla \times \vec{B}\right) \cdot d\vec{a} = \oint_{Loop} \vec{B} \cdot d\vec{l}$$

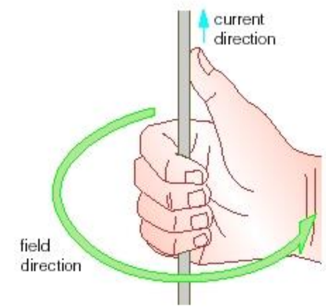


: Stoke's Theorem

: Curl Theorem

The curl measures the "twist" of the vector **v**.



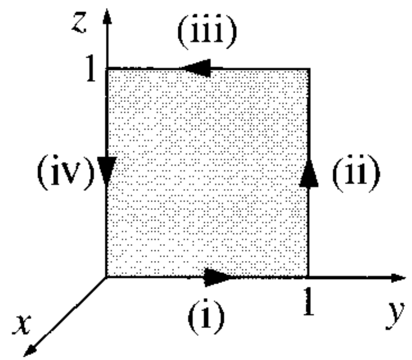


[Corollary 1 (따름 정리1)] $\int \left(\nabla \times \vec{B}\right) \cdot d\vec{a}$ **depends** only on the **boundary line**, not on the particular surface used.

[Corollary 2 (따름 정리2)] $\oint \left(\nabla \times \vec{B}\right) \cdot d\vec{a} = 0$ for any **closed surface**, since the boundary line, like the mouth of a balloon, shrinks down to a point.

1.3 Integral Calculus (적분학)

[Example 1.11] $\vec{B} = \left(2xz + 3y^2\right)\hat{y} + \left(4yz^2\right)\hat{z}$



Check Stoke's theorem for the square surface.

$$\int_S \left(\nabla \times \vec{B}\right)\cdot d\vec{a} = \oint_{Loop} \vec{B}\cdot d\vec{l}$$

$$\nabla \times \vec{B} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \times \left[\left(2xz + 3y^2\right)\hat{y} + \left(4yz^2\right)\hat{z}\right]$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & \left(2xz + 3y^2\right) & \left(4yz^2\right) \end{vmatrix} = \left(4z^2 - 2x\right)\hat{x} + 2z\,\hat{z}$$

$$x = 0;\ d\vec{a} = dy\,dz\,\hat{x},$$

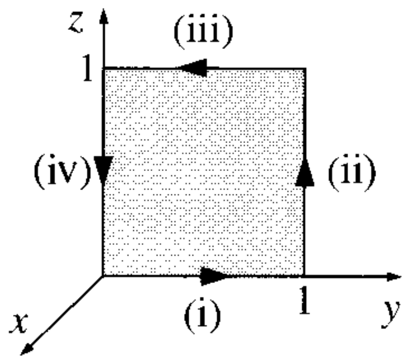
$$\int_S \left(\nabla \times \vec{B}\right)\cdot d\vec{a} = \int_S \left[\left(4z^2 - 2x\right)\hat{x} + 2z\,\hat{z}\right] \cdot \left[dydz\,\hat{x}\right]_{x=0} = \int_S \left(4z^2\right)dydz$$

$$= \int_{z=0}^{1}\left[\int_{y=0}^{1}\left\{4z^2\right\}dy\right]dz = \int_{z=0}^{1}\left[4z^2 y\right]_{y=0}^{1} dz = \int_{z=0}^{1} 4z^2 dz = \frac{4}{3}$$

For the surface(vi)

1.3 Integral Calculus (적분학)

[Example 1.11] *(continued)* $\vec{B} = \left(2xz + 3y^2\right)\hat{y} + \left(4yz^2\right)\hat{z}$



For the line (i) $x = 0,\ z = 0;\ d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} = dy\,\hat{y}$

$$\vec{B}\cdot d\vec{l} = \left[\left(2xz + 3y^2\right)\hat{y} + \left(4yz^2\right)\hat{z}\right]_{x=0,z=0} \cdot dy\,\hat{y}$$

$$= \left(2xz + 3y^2\right)_{x=0,z=0} dy = 3y^2 dy$$

$$\int_{(\mathrm{i})} \vec{B}\cdot d\vec{l} = \int_0^1 3y^2 dy = \left[y^3\right]_0^1 = 1$$

For the line (ii) $x = 0,\ y = 1;\ d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} = dz\,\hat{z}$

$$\vec{B}\cdot d\vec{l} = \left[\left(2xz + 3y^2\right)\hat{y} + \left(4yz^2\right)\hat{z}\right]_{x=0,y=1} \cdot dz\,\hat{z} = 4z^2 dz$$

$$\int_{(\mathrm{ii})} \vec{B}\cdot d\vec{l} = \int_0^1 4z^2 dz = \left[\frac{4}{3}z^3\right]_0^1 = \frac{4}{3}$$

For the line (iii) $x = 0,\ z = 1;\ d\vec{l} = dy\,\hat{y},\qquad \vec{B}\cdot d\vec{l} = 3y^2 dy$

$$\int_{(\mathrm{iii})} \vec{B}\cdot d\vec{l} = \int_1^0 3y^2 dy = \left[y^3\right]_1^0 = -1$$

For the line (iv) $x = 0,\ y = 0;\ d\vec{l} = dz\,\hat{z},\qquad \vec{B}\cdot d\vec{l} = \left(4yz^2\right)dz = 0,\qquad \int_{(\mathrm{iv})} \vec{B}\cdot d\vec{l} = 0$

$$\oint_{Loop} \vec{B}\cdot d\vec{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}$$

1.3 Integral Calculus (적분학)

1.3.6 Integration by Parts (성분별 적분)

$$\frac{d}{dx}(fg) = f\left(\frac{dg}{dx}\right) + g\left(\frac{df}{dx}\right)$$

Integrating both sides:

$$\int_a^b \frac{d}{dx}(fg)dx = fg\Big|_a^b = \int_a^b f\left(\frac{dg}{dx}\right)dx + \int_a^b g\left(\frac{df}{dx}\right)dx$$

or

$$\int_a^b f\left(\frac{dg}{dx}\right)dx = -\int_a^b g\left(\frac{df}{dx}\right)dx + fg\Big|_a^b$$

: **Integration by Parts** (성분별 적분)

[Example 1.12] Evaluate the integral $\int_0^\infty xe^{-x}dx$

$$e^{-x} = \frac{d}{dx}\left(-e^{-x}\right), \qquad f(x) = x, \quad g(x) = -e^{-x}, \quad \frac{df}{dx} = 1$$

$$\int_0^\infty xe^{-x}dx = -\int_0^\infty \left(-e^{-x}\right)dx - xe^{-x}\Big|_{x=0}^{\infty} = \left[-e^{-x}\right]_0^\infty = 1$$

1.3 Integral Calculus (적분학)

Some Integral Relations

$$\nabla \cdot \left(f\vec{A}\right) = f\left(\nabla \cdot \vec{A}\right) + \vec{A} \cdot (\nabla f)$$

Integrating the above equation over a volume and using the divergence theorem, we obtain

$$\int_V \nabla \cdot \left(f\vec{A}\right) dV = \int_V f\left(\nabla \cdot \vec{A}\right) dV + \int_V \vec{A} \cdot (\nabla f) dV = \oint_S f\vec{A} \cdot d\vec{a}$$

or

$$\int_V f\left(\nabla \cdot \vec{A}\right) dV = -\int_V \vec{A} \cdot (\nabla f) dV + \oint_S f\vec{A} \cdot d\vec{a}$$

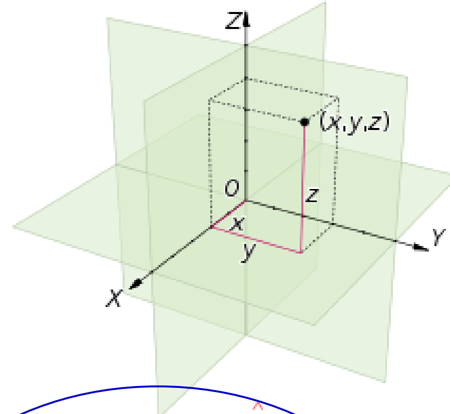
Chapter 1. Vector Analysis

- Vector Algebra
- Differential Calculus
- Integral Calculus
- Curvilinear Coordinates
- The Dirac Delta Function
- The Theory of Vector Fields

1.4 Curvilinear Coordinates (곡선 좌표계)

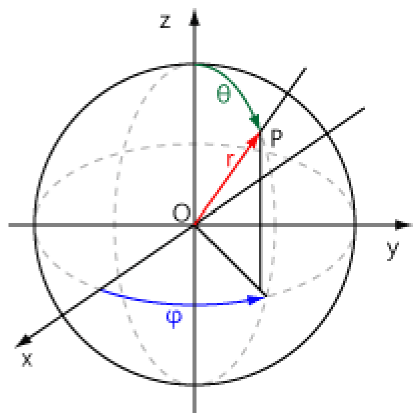
- **Cartesian Coordinates (직교 좌표계)**

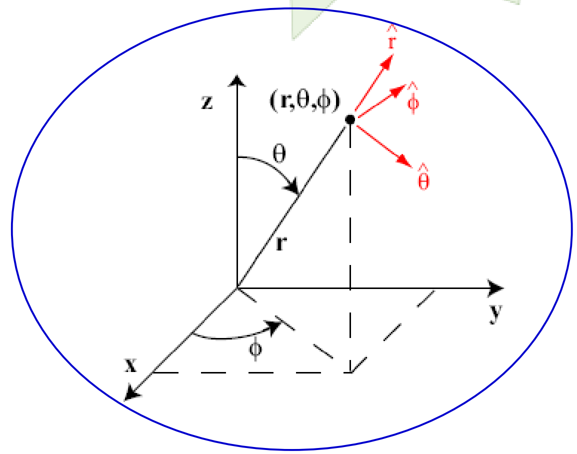
$$(x, y, z)$$

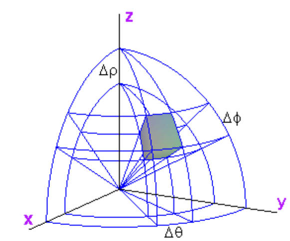


- **Spherical Coordinates (구 좌표계)**

$$(r, \theta, \phi)$$





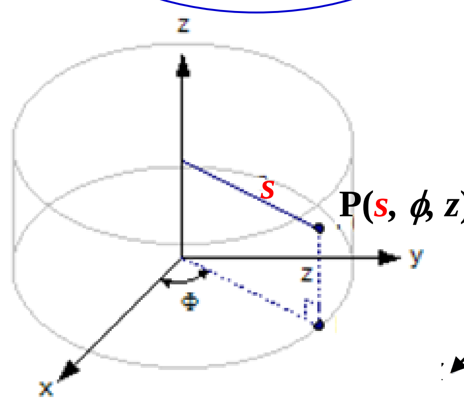


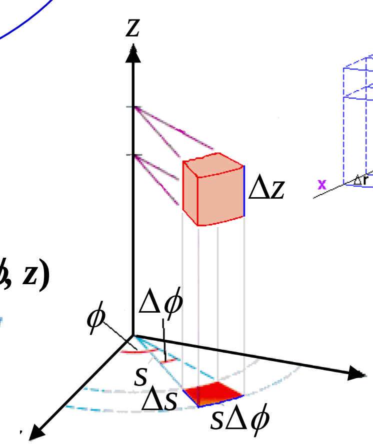
- **Cylindrical Coordinates (원통 좌표계)**

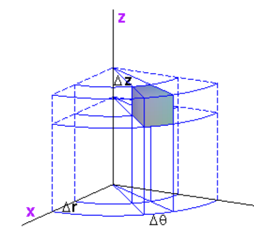
$$(s, \phi, z)$$

(r, ϕ, z) (ρ, ϕ, z) (r, θ, z)

(ρ, θ, z)







1.4 Curvilinear Coordinates (곡선 좌표계)

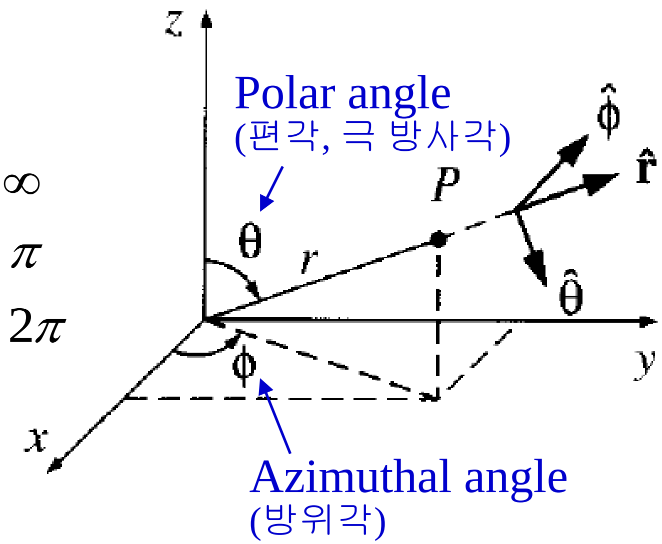
1.4.1 Spherical Coordinates (구 좌표계)

$$(r,\theta,\phi)$$

r : the distance from the origin (원점으로 부터의 거리) $r=0\sim\infty$

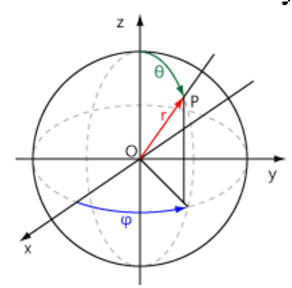
θ : the polar angle (편각, 극 방사각) $\theta=0\sim\pi$

ϕ : the azimuthal angle (방위각) $\phi=0\sim 2\pi$



- Cartesian coordinates 와의 관계

$$x=r\sin\theta\cos\phi$$
$$y=r\sin\theta\sin\phi \qquad (1.62)$$
$$z=r\cos\theta$$



- A vector in the spherical coordinates

$$\vec{A}=A_r\hat{r}+A_\theta\hat{\theta}+A_\phi\hat{\phi} \qquad (1.63)$$

$$\left(\begin{aligned}\hat{r}&=\sin\theta\cos\phi\,\hat{x}+\sin\theta\sin\phi\,\hat{y}+\cos\theta\,\hat{z}\\ \hat{\theta}&=\cos\theta\cos\phi\,\hat{x}+\cos\theta\sin\phi\,\hat{y}-\sin\theta\,\hat{z}\\ \hat{\phi}&=-\sin\phi\,\hat{x}+\cos\phi\,\hat{y}\end{aligned}\right) \qquad (1.64)$$

These are related to a particular point P.

They change direction as P moves around.

1.4 Curvilinear Coordinates (곡선 좌표계)

1.4.1 Spherical Coordinates (구 좌표계) *(continued)*

- Infinitesimal displacement (극소 변위)

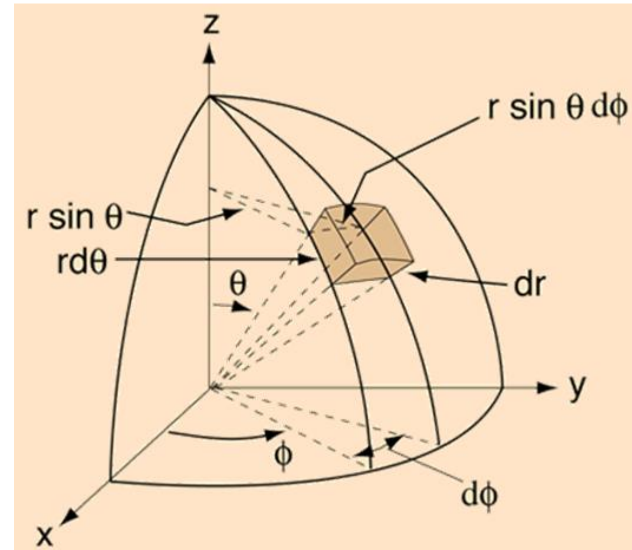
$$dl_r = dr$$

$$dl_\theta = r\,d\theta \qquad (1.66)$$

$$dl_\phi = r\sin\theta\,d\phi \qquad (1.67)$$

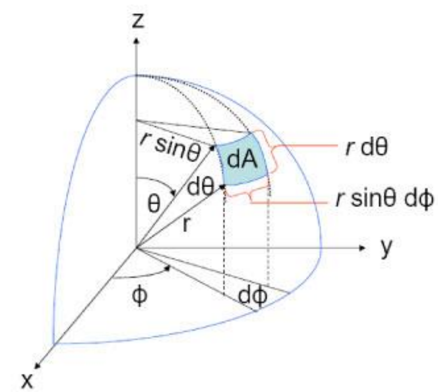
$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi} \qquad (1.68A)$$

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{\partial}{r\,\partial\theta} + \hat{\phi}\frac{\partial}{r\sin\theta\,\partial\phi} \qquad (1.68B)$$



- Infinitesimal volume element (극소 부피체) dV

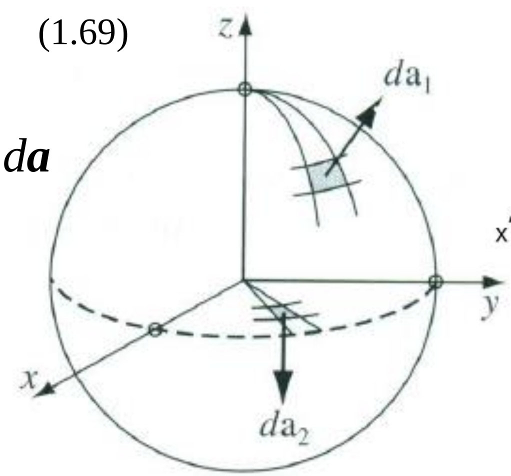
$$dV = dl_r dl_\theta dl_\phi = r^2 \sin\theta\,dr\,d\theta\,d\phi \qquad (1.69)$$



- Infinitesimal surface element (극소 면적판) ***da***

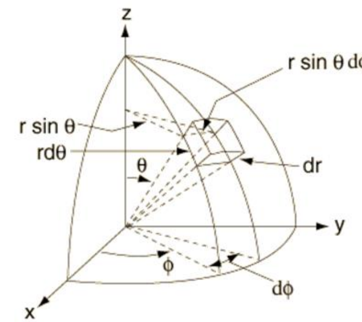
$$d\vec{a}_1 = dl_\theta dl_\phi\,\hat{r} = r^2 \sin\theta\,d\theta\,d\phi\,\hat{r}$$

$$d\vec{a}_2 = dl_r dl_\phi\,\hat{\theta} = r\,dr\,d\phi\,\hat{\theta}$$



1.4 Curvilinear Coordinates (곡선 좌표계)

1.4.1 Spherical Coordinates (구 좌표계)

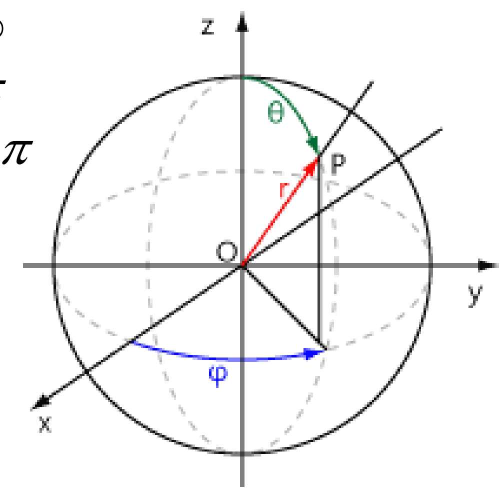


[Example 1.13] Find the volume of a sphere of radius *R*.

$$dV = dl_r dl_\theta dl_\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$r = 0 \sim \infty$$
$$\theta = 0 \sim \pi$$
$$\phi = 0 \sim 2\pi$$



$$V = \int dV = \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \left(\int_{r=0}^{R} r^2\, dr\right)\left(\int_{\theta=0}^{\pi} \sin\theta\, d\theta\right)\left(\int_{\phi=0}^{2\pi} d\phi\right)$$

$$= \left(\frac{R^3}{3}\right)\left[-\cos\theta\right]_{\theta=0}^{\pi}(2\pi) = \left(\frac{R^3}{3}\right)(2)(2\pi) = \frac{4}{3}\pi R^3$$

1.4 Curvilinear Coordinates (곡선 좌표계)

1.4.1 Spherical Coordinates (구 좌표계) *(continued)*

● **Gradient, divergence, curl, and Laplacian in spherical coordinates**

$$\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial T}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\right) + \frac{\partial T}{\partial \phi}\left(\frac{\partial \phi}{\partial x}\right)$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial r}\left(\frac{\partial r}{\partial y}\right) + \frac{\partial T}{\partial \theta}\left(\frac{\partial \theta}{\partial y}\right) + \frac{\partial T}{\partial \phi}\left(\frac{\partial \phi}{\partial y}\right)$$

……

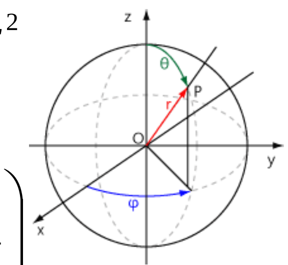
$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$x^2 + y^2 + z^2$$
$$= (r\sin\theta\cos\phi)^2 + (r\sin\theta\sin\phi)^2 + (r\cos\theta)^2$$
$$= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + (r\cos\theta)^2$$
$$= r^2(\sin^2\theta + \cos^2\theta) = r^2$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$



Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi} \qquad (1.70)$$

Divergence:

$$\nabla \cdot \vec{\mathrm{v}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \mathrm{v}_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, \mathrm{v}_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \mathrm{v}_\phi}{\partial \phi} \qquad (1.71)$$

Curl:

$$\nabla \times \vec{\mathrm{v}} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta\, \mathrm{v}_\phi\right) - \frac{\partial \mathrm{v}_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial \mathrm{v}_r}{\partial \phi} - \frac{\partial}{\partial r}\left(r^2 \mathrm{v}_\phi\right)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\, \mathrm{v}_\theta\right) - \frac{\partial \mathrm{v}_r}{\partial \theta}\right]\hat{\phi} \qquad (1.72)$$

[별첨 Ref1 파일 참고]

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} \qquad (1.73)$$

1.4 Curvilinear Coordinates (곡선 좌표계)

1.4.2 Cylindrical Coordinates (원통 좌표계)

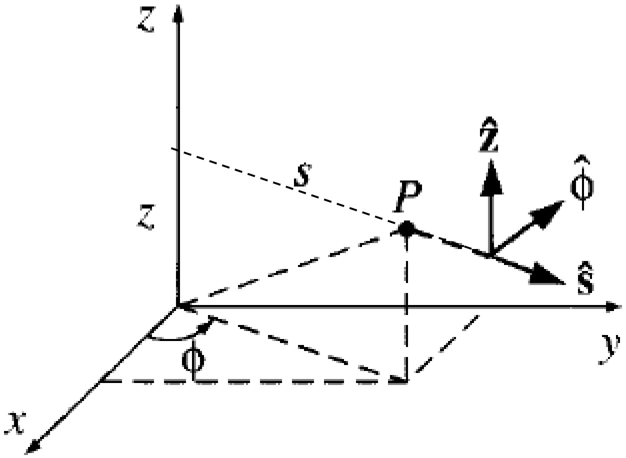
$$\left(s,\phi,z\right)$$

s : the distance from the z axis to the point P

ϕ : the azimuthal angle (방위각)

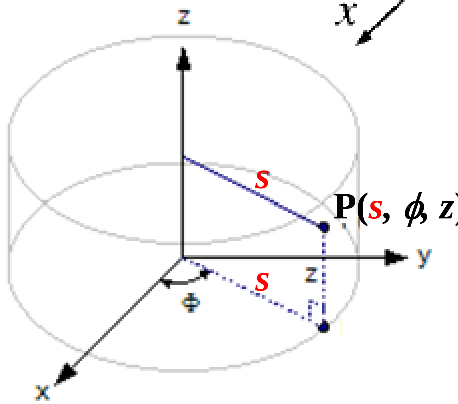
z : the z-axis

$$s = 0 \sim \infty$$
$$\phi = 0 \sim 2\pi$$
$$z = -\infty \sim +\infty$$



- Cartesian coordinates 와의 관계

$$x = s\cos\phi$$
$$y = s\sin\phi \qquad (1.74)$$
$$z = z$$



$$s^2 = x^2 + y^2$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

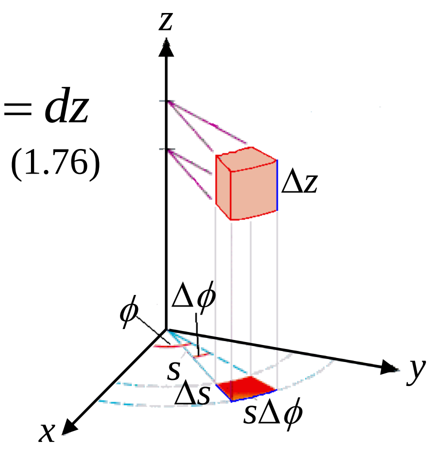
- A vector in the cylindrical coordinates

$$\vec{\mathrm{A}} = \mathrm{A}_s\hat{s} + \mathrm{A}_\varphi\hat{\phi} + \mathrm{A}_z\hat{z}$$

$$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \qquad (1.75)$$
$$\hat{z} = \hat{z}$$

1.4 Curvilinear Coordinates (곡선 좌표계)

1.4.2 Cylindrical Coordinates (원통 좌표계) *(continued)*



- Infinitesimal displacement (극소 변위) $dl_s = ds, \quad dl_\phi = s\, d\phi, \quad dl_z = dz$ (1.76)

$$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z} \qquad (1.77)$$

- Infinitesimal volume element (극소 부피체) dV

$$dV = dl_s dl_\phi dl_z = s\,ds\,d\phi\,dz \qquad (1.78)$$

- **Gradient:**

$$\nabla T = \frac{\partial T}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z} \qquad (1.79)$$

- **Divergence:**

$$\nabla \cdot \vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \qquad (1.80)$$

- **Curl:**

$$\nabla \times \vec{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s\,v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{z} \qquad (1.81)$$

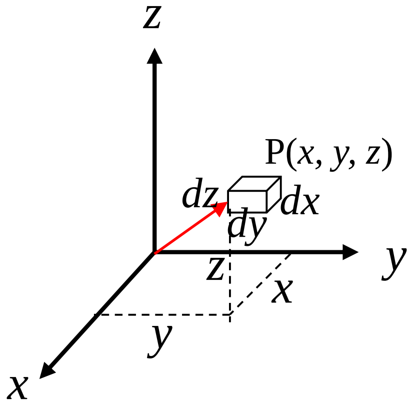
- **Laplacian:**

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \qquad (1.82)$$

[별첨 Ref2 파일 참고]

1.4 Curvilinear Coordinates (곡선 좌표계)

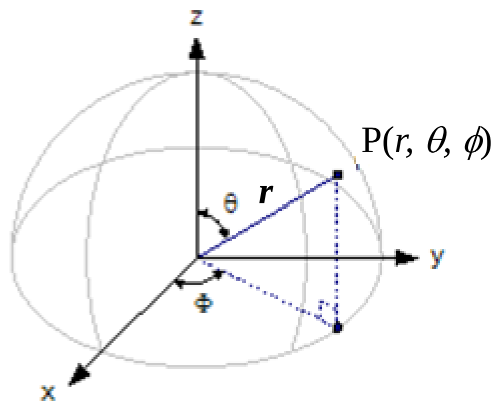
Cartesian (Rectangular) Coordinates



Differential volume =

$$dx\, dy\, dz$$

Spherical Coordinates

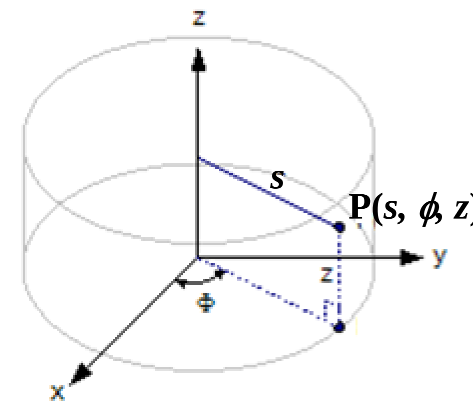


Differential volume =

$$r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$
$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

Cylindrical Coordinates



Differential volume =

$$s^2 ds\, d\phi\, dz$$

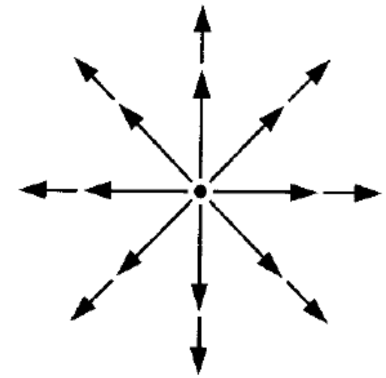
$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$
$$s^2 = x^2 + y^2$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

Chapter 1. Vector Analysis

- Vector Algebra
- Differential Calculus
- Integral Calculus
- Curvilinear Coordinates
- The Dirac Delta Function
- The Theory of Vector Fields

1.5 The Dirac Delta Function (델타 함수)

1.5.1 The Divergence of $\hat{r}/r^2$

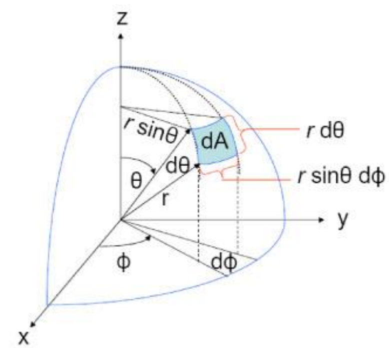


$$\vec{\mathrm{v}} = \frac{1}{r^2}\hat{r} \qquad (1.83)$$

From the divergence of spherical coordinates,

Eq. (1.71) →
$$\nabla\cdot\vec{\mathrm{v}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \mathrm{v}_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\mathrm{v}_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \mathrm{v}_\phi}{\partial\phi}$$

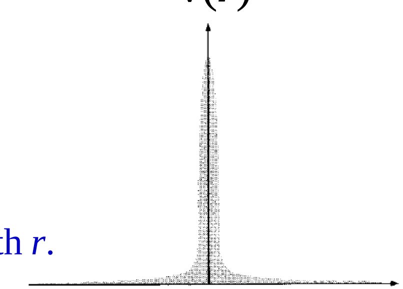
$$\nabla\cdot\vec{\mathrm{v}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0 \qquad (1.84)$$



$$\int_V (\nabla\cdot\vec{\mathrm{v}})\,dV = \int_V 0\,dV = 0$$

$$\oint \vec{\mathrm{v}}\cdot d\vec{a} = \int_S \left(\frac{1}{R^2}\hat{r}\right)\cdot\left(R^2\sin\theta\,d\theta\,d\phi\,\hat{r}\right) = \left(\int_{\theta=0}^{\pi}\sin\theta\,d\theta\right)\left(\int_{\phi=0}^{2\pi}d\phi\right) = 4\pi \qquad (1.85)$$

$$\int_V (\nabla\cdot\vec{\mathrm{v}})\,dV \neq \oint_S \vec{\mathrm{v}}\cdot d\vec{a}$$ **??**



$$\mathrm{v}(r=0) = \frac{1}{r^2} \rightarrow \infty$$

$$\mathrm{v}(r>0) = \frac{1}{r^2} \propto \frac{1}{r^2}$$ decays sharply with r.

1.5 The Dirac Delta Function (델타 함수)

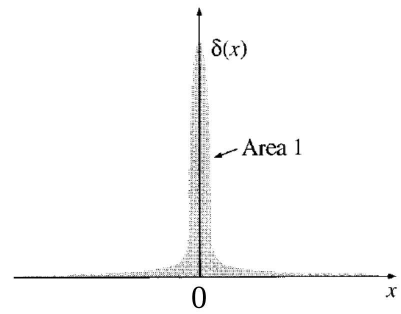
1.5.2 The One-Dimensional Dirac Delta Function

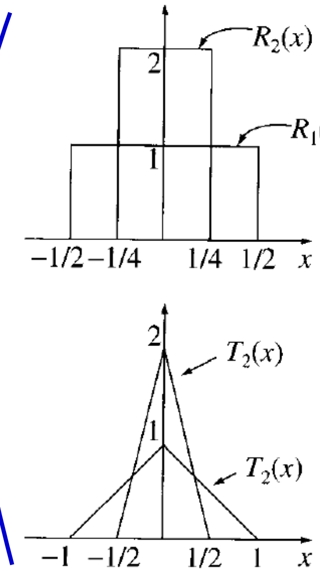
$$\delta(x) = \begin{cases} 0 & if\ x \neq 0 \\ \infty & if\ x = 0 \end{cases} \qquad (1.86)$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \qquad (1.87)$$

: generalized function

: Dirac distribution





Height n
Width $1/n$

Height n
Base $2/n$

For an ordinary function *f*(*x*),

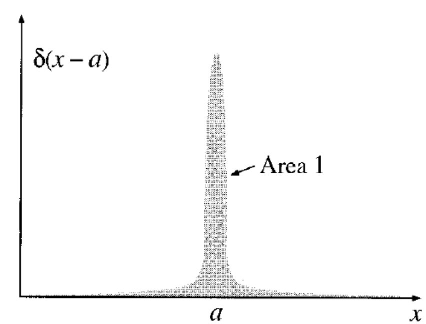
$$f(x)\delta(x) = f(0)\delta(x) \qquad (1.88)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)\int_{-\infty}^{\infty}\delta(x)\,dx = f(0) \qquad (1.89)$$

$$\delta(x-a) = \begin{cases} 0 & if\ x \neq a \\ \infty & if\ x = a \end{cases} \quad \text{with} \quad \int_{-\infty}^{\infty} \delta(x-a)dx = 1 \qquad (1.90)$$

$$f(x)\delta(x-a) = f(a)\delta(x-a) \qquad (1.91)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a) \qquad (1.92)$$



1.5 The Dirac Delta Function (델타 함수)

[Example 1.14] Evaluate the integral $\int_0^3 x^3\delta(x-2)dx.$

(Solution) $\int_0^3 x^3\delta(x-2)dx = 2^3 = 8$

반면에 적분 구간 상한 값이 1일 경우에는 $\int_0^1 x^3\delta(x-2)dx = 0$

[Example 1.15] Show that $\delta(kx) = \dfrac{1}{|k|}\delta(x)$

where k is any (nonzero) constant. $\delta(-x) = \delta(x)$

(Solution) For an arbitrary function $f(x)$, let us consider the integral $\int_{-\infty}^{\infty} f(x)\delta(kx)dx$

$$y \equiv kx \quad \rightarrow \quad x = \frac{y}{k} \quad \rightarrow \quad dx = \frac{1}{k}dy$$

If k is positive, $\int_{-\infty}^{\infty}[\ ]dy$ If k is negative, $\int_{\infty}^{-\infty}[\ ]dy$

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \pm\int_{-\infty}^{\infty} f\left(\frac{y}{k}\right)\delta(y)\frac{dy}{k} = \pm\frac{1}{k}f(0) = \frac{1}{|k|}f(0) = \frac{1}{|k|}\int_{-\infty}^{\infty} f(x)\delta(x)dx$$

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \int_{-\infty}^{\infty} f(x)\left[\frac{\delta(x)}{|k|}\right]dx \longrightarrow \therefore\ \delta(kx) = \frac{1}{|k|}\delta(x)$$

1.5 The Dirac Delta Function (델타 함수)

1.5.3 The Three-Dimensional Delta Function

$$\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z) \qquad (1.96)$$

where $\vec{r} \equiv x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$: a position vector from (0, 0, 0) to (*x*, *y*, *z*)

$$\int_{all\ space} \delta^3(\vec{r})dv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\, dx\, dy\, dz = 1 \qquad (1.97)$$

For an ordinary function *f(r)*,

$$\int_{all\ space} f(\vec{r})\delta^3(\vec{r}-\vec{a})dv = f(\vec{a}) \qquad (1.98)$$

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r}) \qquad (1.99)$$

$$\vec{v} = \frac{1}{r^2}\hat{r}$$

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

$$\int_V (\nabla\cdot\vec{v})dV = \int_V 0\, dV = 0$$

$$\oint \vec{v}\cdot d\vec{a} = \int_S \left(\frac{1}{R^2}\hat{r}\right)\cdot\left(R^2\sin\theta\, d\theta\, d\phi\,\hat{r}\right) = \left(\int_{\theta=0}^{\pi}\sin\theta\, d\theta\right)\left(\int_{\phi=0}^{2\pi}d\phi\right) = 4\pi$$

← Eq. (1.84)

A general expression :

$$\nabla\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}}) \qquad (1.100)$$

for any separation vector (거리벡터) $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

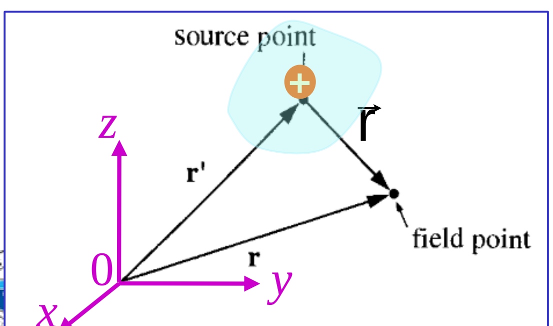
$d\mathscr{r} = dr$ [*r*' is held constant. (*r*' 은 고정)]

$$\nabla = \frac{\partial}{\partial r}\hat{\mathscr{r}}$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \qquad (1.101)$$

←[Appendix C] 참고

$$\nabla^2\left(\frac{1}{\mathscr{r}}\right) = -4\pi\delta^3(\vec{\mathscr{r}}) \qquad (1.102)$$



1.5 The Dirac Delta Function (델타 함수)

[Example 1.16] Evaluate the integral $J = \int_V \left(r^2 + 2\right) \nabla \cdot \left(\frac{\hat{r}}{r^2}\right) dV,$

where V is a sphere of radius R centered at the origin.

(Solution 1) Eq. (1.99) → $\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})$

$$J = \int_V \left(r^2 + 2\right) 4\pi\delta^3(\vec{r})\, d\text{v} = 4\pi\left[\int_V r^2 \delta^3(\vec{r})\, d\text{v} + 2\int_V \delta^3(\vec{r})\, d\text{v}\right]$$

$$= 4\pi(0 + 2) = 8\pi$$

(Solution 2)

Eq. (1.59) → $\int_V f\left(\nabla \cdot \vec{A}\right) d\text{v} = -\int_V \vec{A} \cdot (\nabla f)\, d\text{v} + \oint_S f\vec{A} \cdot d\vec{a}$

$$J = \int_V \left(r^2 + 2\right) \nabla \cdot \left(\frac{\hat{r}}{r^2}\right) d\text{v} = -\int_V \frac{\hat{r}}{r^2} \cdot \nabla\left(r^2 + 2\right) d\text{v} + \oint_S \left(r^2 + 2\right) \frac{\hat{r}}{r^2} \cdot d\vec{a}$$

$$= -\int_V \frac{\hat{r}}{r^2} \cdot 2r\,\hat{r}\, d\text{v} + \oint_S \left(r^2 + 2\right) \frac{\hat{r}}{r^2} \cdot d\vec{a}$$

$\left\langle \nabla\left(r^2 + 2\right) = 2r\,\hat{r} \right\rangle$

$$= -\int_V \frac{2}{r} r^2 \sin\theta\, dr\, d\theta\, d\phi + \left[\oint_S \left(r^2 + 2\right) \frac{\hat{r}}{r^2} \cdot r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}\right]_{r=R}$$

$$= -\int_{r=0}^{R} 2r dr \int_{\theta=0}^{\pi} \sin\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi + \left(R^2 + 2\right) \int_{\theta=0}^{\pi} \sin\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= -4\pi R^2 + 4\pi\left(R^2 + 2\right) = 8\pi$$

1.6 The Theory of Vector Fields (벡터 장 이론)

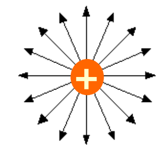
1.6.1 The Helmholtz Theorem

Electric field : $\vec{E}$

Magnetic field : $\vec{B}$

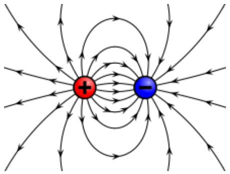
$\left.\begin{matrix}\vec{E}\\ \vec{B}\end{matrix}\right\} \vec{F}$ 우선 편의적으로 전기장과 자기장을 F로 둠.

(*i.e.*, $\vec{F}=\vec{E}$ or $\vec{F}=\vec{B}$)



$$\nabla\cdot\vec{E}=\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}$$

$$\nabla\times\vec{E}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix}$$



$$=\hat{x}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)+\hat{y}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+\hat{z}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)$$

- Divergence: $\nabla\cdot\vec{F}=\underbrace{D}_{\text{Scalar 량}}$

- Curl: $\nabla\times\vec{F}=\vec{C}$

$$\nabla\cdot\vec{C}=\nabla\cdot\left(\nabla\times\vec{F}\right)=0$$ Eq. (1.46) 참조

[Appendix B] $\vec{F}=-\nabla U+\nabla\times\vec{W}$

where $U(\vec{r})\equiv\frac{1}{4\pi}\int\frac{D(\vec{r}\,')}{\mathfrak{r}}dv'$, : a scalar potential

$\vec{W}(\vec{r})\equiv\frac{1}{4\pi}\int\frac{\vec{C}(\vec{r}\,')}{\mathfrak{r}}dv'$: a vector potential

$$\nabla\cdot\vec{F}=-\nabla^2U=-\frac{1}{4\pi}\int D\nabla^2\left(\frac{1}{\mathfrak{r}}\right)dv'=\int D(\vec{r}\,')\delta^3(\vec{r}-\vec{r}\,')dv'=D(\vec{r})$$

$$\nabla\times\vec{F}=\nabla\times\left(\nabla\times\vec{W}\right)=-\nabla^2\vec{W}+\nabla\left(\nabla\cdot\vec{W}\right)$$

$$-\nabla^2\vec{W}=-\frac{1}{4\pi}\int\vec{C}\nabla^2\left(\frac{1}{\mathfrak{r}}\right)dv'=\int\vec{C}(\vec{r}\,')\delta^3(\vec{r}-\vec{r}\,')dv'=\vec{C}(\vec{r})$$

$$\left(\nabla\cdot\left(\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right)=4\pi\delta^3(\mathfrak{r})\right.$$

$$\left.\nabla\left(\frac{1}{\mathfrak{r}}\right)=-\frac{\hat{\mathfrak{r}}}{\mathfrak{r}^2}\right)$$

1.6 The Theory of Vector Fields (벡터 장 이론)

1.6.2 Potentials

- Curl of a vector field: $\nabla \times \vec{F} = 0$ ⟷ $\vec{F} = -\nabla V$ (a scalar potential)

[Theorem 1] Curl-less (or "irrotational") fields

(a) $\nabla \times \vec{F} = 0$ everywhere

(b) $\int_a^b \vec{F} \cdot d\vec{l}$ is independent of path, for any given end points.

(c) $\oint \vec{F} \cdot d\vec{l} = 0$ for any closed loop.

(d) $\vec{F}$ is the gradient of some scalar function $\vec{F} = -\nabla V$

- Curl of a vector field: $\nabla \cdot \vec{F} = 0$ ⟷ $\vec{F} = \nabla \times \vec{A}$ (a vector potential) ⟶ $\vec{F} = -\nabla V + \nabla \times \vec{A}$

[Theorem 2] Divergence-less (or "solenoidal") fields

(a) $\nabla \cdot \vec{F} = 0$ everywhere

(b) $\int \vec{F} \cdot d\vec{a}$ is independent of surface, for any given boundary line.

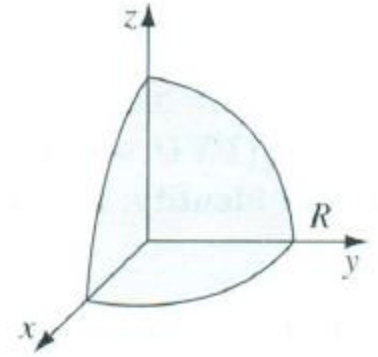
(c) $\oint \vec{F} \cdot d\vec{a} = 0$ for any closed surface.

(d) $\vec{F}$ is the curl of some vector function $\vec{F} = -\nabla \times \vec{A}$

[Problem 1.54] Check the divergence theorem for the function

$$\mathbf{v} = r^2\cos\theta\,\hat{\mathbf{r}} + r^2\cos\phi\,\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}},$$

using as your volume one octant of the sphere of radius R (Fig. 1.48). Make sure you include the *entire* surface. [*Answer:* $\pi R^4/4$]



(Solution)

$$\int_V (\nabla\cdot\vec{\mathrm{v}})d\mathrm{v} = \oint_S \vec{\mathrm{v}}\cdot d\vec{a}$$

$$\nabla\cdot\vec{\mathrm{v}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^2\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, r^2\cos\phi\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-r^2\cos\theta\sin\phi\right)$$

$$= \frac{1}{r^2}4r^3\cos\theta + \frac{1}{r\sin\theta}\cos\theta\, r^2\cos\phi + \frac{1}{r\sin\theta}\left(-r^2\cos\theta\cos\phi\right)$$

$$= \frac{r\cos\theta}{\sin\theta}\left[4\sin\theta + \cos\phi - \cos\phi\right] = 4r\cos\theta.$$

$$\int_V(\nabla\cdot\vec{\mathrm{v}})d\mathrm{v} = \int_V (4r\cos\theta)r^2\sin\theta\,dr\,d\theta\,d\phi = 4\int_{r=0}^{R} r^3 dr\int_{\theta=0}^{\pi/2}\cos\theta\sin\theta\,d\theta\int_{\phi=0}^{\pi/2}d\phi$$

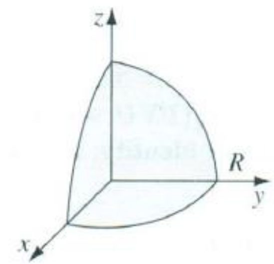
$$= R^4\,\frac{1}{2}\,\frac{\pi}{2} = \frac{\pi R^4}{4}$$

Problem 1.54

[Problem 1.54]

$$\vec{v} = r^2 \cos\theta\, \hat{r} + r^2 \cos\phi\, \hat{\theta} - r^2 \cos\theta \sin\phi\, \hat{\phi}$$

(Solution) - *continued* $\oint_S \vec{v} \cdot d\vec{a}$ for 4 surfaces



i) Curved surface: $r = R, \quad d\vec{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

$$\vec{v} \cdot d\vec{a} = \left(R^2 \cos\theta\right)\left(R^2 \sin\theta\, d\theta\, d\phi\right) = R^4 \cos\theta \sin\theta\, d\theta\, d\phi$$

$$\int_{(i)} \vec{v} \cdot d\vec{a} = R^4 \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta \int_0^{\pi/2} d\phi = R^4 \left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi R^4}{4}$$

ii) Left surface: $\phi = 0, \quad d\vec{a} = -r d\theta\, dr\, \hat{\phi}, \quad \vec{v} \cdot d\vec{a} = \left(-r^2 \cos\theta \sin\phi\right)\left(-r\, dr\, d\theta\right) = 0$

$$\int_{(ii)} \vec{v} \cdot d\vec{a} = \sin\phi \int_0^R r^3 dr \int_0^{\pi/2} \cos\theta\, d\theta = 0$$

iii) Back surface: $\phi = \pi/2, \quad d\vec{a} = r d\theta\, dr\, \hat{\phi}$

$$\vec{v} \cdot d\vec{a} = \left(-r^2 \cos\theta \sin\phi\right)\left(r\, dr\, d\theta\right) = -r^3 \cos\theta\, dr\, d\theta$$

$$\int_{(iii)} \vec{v} \cdot d\vec{a} = -\int_0^R r^3 dr \int_0^{\pi/2} \cos\theta\, d\theta = -\frac{R^4}{4}(1) = -\frac{R^4}{4}$$

iv) Bottom surface: $\theta = \pi/2, \quad d\vec{a} = r d\phi\, dr\, \hat{\theta}, \quad \vec{v} \cdot d\vec{a} = \left(r^2 \cos\phi\right)\left(r\, dr\, d\phi\right) = r^3 \cos\phi\, dr\, d\phi$

$$\int_{(iv)} \vec{v} \cdot d\vec{a} = \int_0^R r^3 dr \int_0^{\pi/2} \cos\phi\, d\phi = \frac{R^4}{4}(1) = \frac{R^4}{4}$$

$$\therefore \oint_S \vec{v} \cdot d\vec{a} = \frac{\pi R^4}{4} + 0 - \frac{R^4}{4} + \frac{R^4}{4} = \frac{\pi R^4}{4}$$

Next Class

- Chapter 2. Electrostatics

Appendix C

Proof of $\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$

$$\mathscr{r} = \left|\vec{r} - \vec{r}\,'\right| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\nabla\left(\frac{1}{\mathscr{r}}\right) = \frac{\partial}{\partial x}\left\{(x-x')^2 + (y-y')^2 + (z-z')^2\right\}^{-1/2}\hat{x} + \frac{\partial}{\partial y}\left\{(x-x')^2 + (y-y')^2 + (z-z')^2\right\}^{-1/2}\hat{y}$$

$$+ \frac{\partial}{\partial z}\left\{(x-x')^2 + (y-y')^2 + (z-z')^2\right\}^{-1/2}\hat{z}$$

$$= -\frac{1}{2}2(x-x')\left\{(x-x')^2 + (y-y')^2 + (z-z')^2\right\}^{-3/2}\hat{x} - \frac{1}{2}2(y-y')\left\{(x-x')^2 + (y-y')^2 + (z-z')^2\right\}^{-3/2}\hat{y}$$

$$-\frac{1}{2}2(z-z')\left\{(x-x')^2 + (y-y')^2 + (z-z')^2\right\}^{-3/2}\hat{z}$$

$$= -\left[(x-x')\,\hat{x} + (y-y')\,\hat{y} + (z-z')\,\hat{z}\right]\left\{(x-x')^2 + (y-y')^2 + (z-z')^2\right\}^{-3/2}$$

$$= -\vec{\mathscr{r}}\frac{1}{\mathscr{r}^3} = -\left(\mathscr{r}\,\hat{\mathscr{r}}\right)\frac{1}{\mathscr{r}^3} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

Back